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Mathematical Reviews

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FOUNDATIONS, THEORY OF SETS, LOGIC

★**Hausdorff, Felix. Set theory.** Translated by John R. Aumann, et al. Chelsea Publishing Company, New York, 1957. 352 pp. $6.00.

A translation of the third edition of 1937 [de Gruyter, Berlin-Leipzig].

★**Lorenzen, Paul. Die Fiktion der Überabzählbarkeit.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 273–279. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

The author starts from "figures" composed of a finite number of "atoms". The notion of set is defined by means of propositions. In order to avoid infinite regressions the author attempts to use "ungestufte Zweige" instead of ramified types of B. Russell. In particular, taking the "row" S_0 as the set of positive integers and using (1) inductive definitions, (2) compositions of logical constants $\rightarrow, \wedge, \vee, \vee_x, \wedge_x, \neg$, one gets by iterations the next row S_1; after that, similarly, one gets $S_2, S_3, \cdots, S_\omega = \bigcup_n S_n$, etc., in an obvious way. The uncountability of the system of "all" subsets of S_0 is interpreted in such a way that $S_0 \subset S_1 \subset \cdots$ and that no row contains all subsets of S_0, because operationally one does not reach, e.g., S_{ω_1}. To show the relative character of "uncountability" which fundamentally occurs, e.g. in topology and integration theory, the author considers two limit ordinals $\theta_1 < \theta_2 < \omega_1$. A row is called primary or secondary according as the corresponding index is $\leqq \theta_1$ or in $(\theta_1, \theta_2]$. Analogously, the sets are divided into primary and secondary ones. Some analogies between countable-uncountable and primary-secondary are illustrated. According to the author, already in the case $\theta_1 = \omega$, $\theta_2 = \omega^\omega$ one gets plenty of propositions about primary-secondary corresponding to propositions insolving countable-uncountable.

Đ. Kurepa (Zagreb).

★**Barkley Rosser, J. The relative strength of Zermelo's set theory and Quine's new foundations.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 289–294. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

Bericht über neuere Arbeiten aus der Schule des Verfassers über die Möglichkeiten innerhalb des Quineschen Systems NF Modelle für die nach Zermelo axiomatisierten Mengenlehren zu finden. Verf. betont, dass noch keine Übersicht über diese Möglichkeiten erreicht wurde. Ein Weg ist die Beschränkung in NF auf "Cantorsche" Mengen, d.h. solche Mengen α, für die eine Zuordnung zwischen den Elementen x von α und den Untermengen $\{x\}$ von α existiert. Die Gödelsche Methode, wohlgeordnete Modelle für die Zermeloschen Mengenlehren zu konstruieren, gestattet dann durch Hinzufügung zu NF von Existenzaxiomen über „Cantorsche" Mengen hoher Kardinalzahlen Modelle für die Zermeloschen Mengenlehren mit entsprechende Existenzaxiomen zu finden.

P. Lorenzen (Kiel).

Fraïssé, Roland. Sur quelques classifications des relations, basées sur des isomorphismes restreints. II. Application aux relations d'ordre, et construction d'exemples montrant que ces classifications sont distinctes. Publ. Sci. Univ. Alger. Sér. A. **2** (1955), 273–295 (1957).

Suite d'un article antérieure [mêmes Publ. **2** (1955), 15–60; MR **18**, 139]. Cette fois l'A. applique aux ensembles ordonnés les notions introduites antérieurement. En particulier, l'A. prouve que la notion de $\underset{p}{\div}$-parenté est consistente relativement à l'addition et la multiplication. Autrement dit, α, α', β, β' étant types d'ordre total, si $\alpha r \alpha'$, $\beta r \beta'$, alors $(\alpha+\beta)\, r\, (\alpha'+\beta')$, $(\alpha\beta)\, r\, (\alpha'\beta')$. Pour l'addition l'énoncé se généralise en considérant la somme ordinale de n'importe quelle famille de types d'ordre total. La lettre r y désigne $\underset{p}{\div}$, $\overset{p}{\div}$, $\overset{\omega}{\div}$. Si $\alpha < \omega^\omega$, $\alpha' \overset{\omega}{\div} \alpha$, alors $\alpha' = \alpha$. Si $\alpha = \lambda\omega^\omega + \rho$, $\lambda \neq 0$, $\rho < \omega^\omega$, $\alpha' \overset{\omega}{\div} \alpha$, alors $\alpha' = \lambda'\omega^\omega + \rho$, λ' designant tout ordinal $\neq 0$. Les deux théorèmes correspondent à deux théorèmes de logique mathématique dûs à Beth [Applications scientifiques de la logique mathématique, Gauthier-Villars, Paris, 1954, pp. 29–35; MR **16**, 556] et Mostowski et Tarski [Bull. Amer. Math. Soc. **55** (1949), 65] respectivement. L'A. montre que les parentés $\overset{p}{\div}$ sont deux à deux distinctes. Par exemple la relation $\omega^n \overset{p}{\div} \omega^{n+1}$ $(n \geqq 1)$ est vérifié pour $p = 2n$ mais non pas pour $p = 2n+1$.

Đ. Kurepa (Zagreb).

Henkin, Léon. The nominalistic interpretation of mathematical language. Bull. Soc. Math. Belg. **7** (1955), 137–142.

This paper is concerned with the interpretation of the nominalistic point of view in mathematics. More particularly, it is suggested that from this point of view class variables are used in a sentence merely in order to indicate the possibility of the substitution of constant predicates, as delimited by an auxiliary language; while classes of classes (of predicates) are concerned with the symbols of these predicates as argument values rather than with the classes themselves (since the latter do not exist, properly speaking, from the nominalistic point of view).

A. Robinson (Toronto, Ont.).

Rasiowa, H. Algebraic models of axiomatic theories. Fund. Math. **41**, 291–310 (1955).

The first part of this paper gives an algebraic definition of general models and proofs of various theorems about these. The most important new results are contained in the second part which deals with axiomatic theories based on the Heyting functional calculus $\mathscr{S}_X{}^*$. A theory $\mathscr{T}$ is called constructive whenever i) if $\sum_{x_p} \alpha$ is a theorem of $\mathscr{T}$, then there exists a term ξ such that the formula which results from α by the substitution of ξ for x_p is a theorem in $\mathscr{T}$; ii) if $\alpha + \beta$ is a theorem in $\mathscr{T}$, then either α or β is a theorem in $\mathscr{T}$. The author then proves that a theory $\mathscr{T}$ is constructive whenever all the axioms of $\mathscr{T}$ (except the axioms of equality) belong to the least set Z^0 of formulas

of $\mathcal{T}$ such that: 1) Z^0 contains all the elementary formulas of $\mathcal{T}$, 2) if $\beta, \gamma \in Z^0$, then $\beta \cdot \gamma \in Z^0$, 3) if $\gamma \in Z^0$, then $\beta \rightarrow \gamma, \neg\beta, \prod_x, \gamma \in Z^0$. It then follows that the theories of groups, rings and Boolean algebras are constructive.

L. N. Gál (Ithaca, N.Y.).

Rasiowa, H. A proof of ε-theorems. Bull. Acad. Polon. Sci. Cl. III. **3**, 299–302 (1955).

This paper contains an outline of a new algebraic proof of the two ε-theorems of Hilbert and Bernays [Grundlagen der Mathematik, vol. II, Springer, Berlin, 1939]. The method used is the same as in the paper reviewed above.

L. N. Gál (Ithaca, N.Y.).

Quine, W. V. Unification of universes in set theory. J. Symb. Logic **21** (1956), 267–279.

A theory is called standardized if it is formulated within quantification theory with only one kind of variable ranging over a single universe and with the non-logical constants meaningful whenever applied to the correct number of variables. As all the known results of the first order predicate calculus apply to such theories, it is useful to put them in this form. This is easily done for Bernays set-theory, and Quine here also carries it out for the theory of types, where normally there is a separate universe for each type. He does it without introducing any new predicate (one for each type), leaving 'ϵ' as the only non-logical connective. The paper ends with a discussion of the standardization of the theory of classes as based on Frege's functional abstraction.

L. N. Gál (Ithaca, N.Y.).

Halmos, Paul R. The basic concepts of algebraic logic. Amer. Math. Monthly **63** (1956), 363–387.

In addition to furnishing an over-all view of the recent developments in algebraic logic, this (relatively) non-technical paper will serve as an excellent introduction to the larger papers by the author [Compositio Math. **12** (1956), 217–249; MR **17**, 1172; see also the two reviews below]. These papers start at the level of the monadic Boolean algebra, i.e., the algebraic equivalent of the first order monadic predicate calculus. The present paper provides a more elementary discussion of Boolean algebra and explains in what sense it is the algebraization of the classical propositional calculus. The emphasis throughout the paper is on motivation, and to the reader who has a knowledge of concepts of modern algebra, such as group, ring, ideal, etc., but who is less well acquainted with the terminology of symbolic logic, the frequent analogies with concepts and methods of group theory should prove helpful and enlightening. The paper falls naturally into three parts. In the first part the propositional calculus is introduced as a collection of equivalence classes obtained by endowing a set S (of sentences) with an equivalence relation, these equivalence classes being naturally isomorphic to the free Boolean algebra generated by the elements of S. Quotient algebras, ideals, filters etc. are then brought in through discussions of their logical analogues; e.g., a Boolean logic is a pair (A, M), where A is a Boolean algebra (whose elements are called propositions) and M is a Boolean ideal (the refutable propositions). A Boolean logic is called consistent if for no p in A are both p and p' in M, and it is complete if for each p either p or p' is in M. It is then argued that a necessary and sufficient condition that (A, M) be consistent and complete is that $A/M = \{0,1\}$. The second part of the paper, which deals with quantifiers and monadic Boolean algebras, lays the foundation for the discussion of polyadic algebras which follows. A monadic algebra is a pair $(A, \exists)$, where A is a Boolean algebra and $\exists$ is a mapping of A into itself satisfying certain conditions suggested by the existential quantifiers of the first-order predicate calculus. By introducing the concept of a constant (as a Boolean endomorphism c of A such that $c\exists=\exists$ and $\exists c=c$) the author shows that monadic algebras have enough structure to contain all of Aristotelian logic. The need for propositional functions of more than one variable, and therefore quantification on several variables and substitution of variables, leads to the definition of polyadic algebras in the third part of the paper. Here semantic concepts are also introduced, such as validity, satisfiability, semantic consistency and completeness, etc. A model is a particular species of polyadic algebra, while an interpretation is a homomorphic mapping of a polyadic algebra into a model, a distinction not always made by logicians. The paper concludes with a discussion of the Gödel completeness and incompleteness theorems. The completeness theorem is translated, for example, into the statement that every locally finite polyadic algebra of infinite degree is semi-simple. (The theorem thus formulated is proved in the second of the above-mentioned papers by the author.)

B. A. Galler (Ann Arbor, Mich.).

Halmos, P. R. Algebraic logic. II. Homogeneous locally finite polyadic Boolean algebras of infinite degree. Fund. Math. **43** (1956), 255–325.

This paper represents the culmination of the author's work on the algebraic-axiomatic version of the lower predicate calculus. Its principal purpose is the proof of a counterpart of the (extended) completeness theorem of Gödel and others. In the axiomatic version, the concept of a polyadic algebra takes the place of an (applied) lower predicate calculus. An intermediate position is occupied by the functional polyadic algebras, which are Boolean sub-algebras of functions from cartesian powers of a given set into a given Boolean algebra, these sub-algebras being closed with respect to substitution and quantification. It is shown that every locally finite polyadic algebra of infinite degree is isomorphic to a functional polyadic algebra. (In this connection, the condition that it be of infinite degree though locally finite corresponds to the classical requirement that the total number of variables be infinite while each predicate depends only on a finite number of variables.) This result is strengthened so as to show that if the given polyadic algebra is simple then it is isomorphic to a functional polyadic algebra which takes values in the two-element Boolean algebra. Moreover, this functional polyadic algebra can be embedded in one which contains a sufficient number of constants to make it correspond to the classical concept of a model. Taken together with the fact that a polyadic algebra is semi-simple, these statements amount to the above mentioned completeness theorem. The usual limits on the cardinality of the models are also obtained here. — Some corrigenda and addenda to part I [Compositio Math. **12** (1956), 217–249; MR **17**, 1172] are listed at the end of the paper. — In his review of L. Henkin, La structure algébrique des théories mathématiques [Gauthier-Villars, Paris, 1956; MR **18**, 272], the present reviewer stressed the need for a comparison between the theories of polyadic and ofcy lindrical algebras. In the meantime, such a comparison has been carried out by B. A. Galler [see the review second below].

A. Robinson.

Halmos, Paul R. Algebraic logic. III. Predicates, terms, and operations in polyadic algebras. Trans. Amer. Math. Soc. 83 (1956), 430–470.

This paper continues the author's development of an axiomatic version of the predicate calculus. Thus, the concepts necessary for quantification (more particularly, for existential quantification) are defined axiomatically with reference to a Boolean algebra, the result being a "polyadic algebra". Since by its very character the syntactical method by which terms, well-formed formulae, predicates, substitution, etc., are defined progressively (inductively) in the traditional way is not available, it becomes necessary to extract these concepts by different, and rather more arduous, methods from the algebraic notions in terms of which a polyadic algebra is defined. The properties of these algebraic counterparts of terms, etc., are developed and discussed here in great detail. According to a statement by the author, this is done with a view to providing a proof of Gödel's incompleteness theorem within the framework of the theory of polyadic algebras. *A. Robinson* (Toronto, Ont.).

Galler, Bernard A. Cylindric and polyadic algebras. Proc. Amer. Math. Soc. 8 (1957), 176–183.

Die Formeln $p, q, \cdots$ des elementaren Prädikatenkalküls bilden nicht nur einen Booleschen Verband B, sie liefern vielmehr noch eine Menge I von Variablen $i, j, \cdots$ und durch die Existenzquantoren Abbildungen $C(i)$, $C(j), \cdots$ von B in sich. Die Gleichheitsformeln $i=j$ liefern schliesslich eine Abbildung d von $I\times I$ in B. Tarski und Thompson [Bull. Amer. Math. Soc. 58 (1952), 65] haben ein Axiomensystem aufgestellt, das von (B, I, C, d) erfüllt wird. Die Modelle dieses Systems heissen "zylindrische" Algebren. Mehrfache Quantoren $\exists(j_1, \cdots, j_n)$ können durch $C(j_1)C(j_2)\cdots C(j_n)$ definiert werden, Substitutionen durch $S(i/j)p=C(i)[p\wedge d(i, j)]$, entsprechend simultane Substitutionen. Halmos [Proc. Nat. Acad. Sci. U.S.A. 40 (1954), 296–301; MR 15, 771] hat ein Axiomensystem für $(B, I, S, \exists)$ aufgestellt. Die Modelle dieses Systems heissen "polyadische" Algebren. Verf. erweitert das Halmossche System um die Axiome (1) $d(j, j)=1$, (2) $p\wedge d(i, j)\leq S(i/j)p$ und beweist anschliessend die Gleichwertigkeit der so erhaltenen "polyadischen Algebren mit Gleichheit" mit den zylindrischen Algebren. *P. Lorenzen* (Kiel).

Copeland, A. H., Sr. Note on cylindric algebras and polyadic algebras. Michigan Math. J. 3 (1955–1956), 155–157.

Die Formeln $x, y, \cdots$ einer mathematischen Theorie A bilden bezüglich $\wedge, \vee, \neg$ eine Boolesche Algebra. Kommen freie Variable $\xi, \cdots$ in den Formeln vor, z.B. $x(\xi)$, so gibt es noch die Quantoren $\mathbf{V}_\xi, \cdots$ und die Substitutionsoperatoren, bezüglich derer A eine "polyadische Boolesche Algebra" [P. Halmos, Proc. Nat. Acad. Sci. U.S.A. 40 (1954), 296–301; MR 15, 771] ist. Um die Operatoren auf endlich viele zu reduzieren, nimmt Verf. an, die freien Variablen seien gegeben als ξ_ν ($\nu=0, \pm1, \pm2, \cdots$). Es genügen dann 6 Operatoren: $\exists x$ für $\mathbf{V}_{\xi_0}x$, Qx für $x\wedge \xi_0=\xi_1$, $Px(\xi_0, \xi_1)$ für $x(\xi_1, \xi_0)$, Ix für x, $T^{\pm1}x(\xi_{\nu_1}, \cdots, \xi_{\nu_n})$ für $x(\xi_{\nu_1\pm1}, \cdots, \xi_{\nu_n\pm1})$. Für diese wird ein Axiomensystem aufgestellt. *P. Lorenzen* (Kiel).

See also: Adyan, p. 117; Popovici, p. 235; Lee and Chen, p. 235.

ALGEBRA

Combinatorial Analysis

Ford, G. W.; and Uhlenbeck, G. E. Combinatorial problems in the theory of graphs. IV. Proc. Nat. Acad. Sci. U.S.A. 43 (1957), 163–167.

In part III [same Proc. 42 (1956), 529–535; MR 18, 326] asymptotic expressions for the number of different star trees with a large number of points built out of stars of a finite number of different types were given. In the present paper it is shown that some restriction on the stars out of which the star trees are constructed has to be made when deriving such an asymptotic expression; if none is made the counting series diverges. It is proved that one restriction which suffices is that each star in the collection has no more than a fixed number of independent cycles. No necessary conditions for the results of Paper III to be valid are given. In the last section the symmetry number of connected star trees built out of stars with not more than c cycles is considered, and it is shown that if $\overline{1/s}$ is the average of the reciprocal of the number of symmetries over the star trees, with p unlabeled points, then, for large p,

$$\overline{1/s}\sim\frac{ab}{3x_0\delta}\left(\frac{\xi_0}{x_0}\right)^{p-5/2}.$$

G. A. Dirac (Vienna).

Bankier, J. D. Generalizations of Pascal's triangle. Amer. Math. Monthly 64 (1957), 416–419.

See also: Parker, p. 163.

Elementary Algebra

Zeckendorf, E. Les suites récurrentes adjointes. Bull. Soc. Roy. Sci. Liège 25 (1956), 636–646.

Properties of the linear second order recurrence $t_x = nt_{x-1}+mt_{x-2}$ with $x=0, \pm1, \pm2, \cdots$ and $t_0=a$, $t_1=b$ are examined by introducing an "adjoint" τ_x defined by $\tau_x=t_{x+1}+mt_{x-1}$ and its iterates,

$$\tau_x(r)=(E+mE^{-1})^r t_x,\quad Et_x=t_{x+1}.$$

Since $\tau_x(2)=(n^2+4m)t_x$, it follows that

$$(n^2+4m)^s t_x=(E+mE^{-1})^{2s}t_x.$$

Also, it is noted that $t_x^2-t_{x-1}t_{x+1}=(-m)^{x-1}(t_1^2-t_0t_2)$, giving these recurrences a similarity to the Fibonacci numbers, and that

$$\tau_1^2-\tau_0\tau_2=-(n^2+4m)(t_1^2-t_0t_2).$$

Some remarks are made on the relation of the parity of the numbers generated by the two recurrences. *J. Riordan.*

Andrade Guimarães, António. A problem in vector calculus. An. Fac. Ci. Porto 37 (1953), 204–209. (Portuguese)

A proof of an elementary result in vector calculus. *E. Lluis* (México, D.F.).

Linear Algebra

Karrer, Guido. Spektraltheorie der Automorphismen Hermite'scher Formen. Ann. Acad. Sci. Fenn. Ser. A. I. no. **237** (1957), 36 pp.

The results of this paper are all special cases of much more general results obtained by H. Jacobinski [Kungl. Fysiogr. Sällsk. Lund Förh. **19** (1949), no. 8; MR **11**, 85]. The author was apparently unaware of the existence of Jacobinski's paper, which he does not mention in his bibliography. *J. Dieudonné* (Evanston, Ill.).

★**McRae, Vincent V. On the unitary similarity of matrices.** Dissertation. The Catholic University of America Press, Washington, D. C., 1955. vii+112 pp.

This thesis deals very thoroughly with the problem of deciding whether two given $n\times n$ matrices over the complex field are similar under the unitary group. This can certainly be accomplished if a canonical set is known, that is, a set of matrices which contains exactly one representative of every unitary equivalence class. The author derives several such canonical sets. Each such set is based on a mode of presenting an arbitrary matrix as a polynomial in (not necessarily commuting) normal matrices. Thus we might choose the presentation $A=N_1+N_2$, where $N_1=(A+A^*)/2, N_2=(A-A^*)/2i$ and where A^* denotes the conjugate complex transpose of A. Or again, we might use the polar form $A=HU$, where H is positive semi-definite and U is unitary. Briefly, the construction proceeds by transforming the constituent normal matrices into canonical form, observing a certain order between them, which is laid down in advance. Thus, the resulting canonical set depends both on the mode of presentation and on the order relation chosen. The work is related to that of J. Brenner on the same problem [Acta Math. **86** (1951), 297–308; MR **13**, 717]. An appendix contains illustrative examples. *W. Ledermann.*

Guttman, Louis. A necessary and sufficient formula for matric factoring. Psychometrika **22** (1957), 79–81.

The following matrix problem occurring in factor analysis is studied. Let A be an arbitrary (real) $p\times q$ matrix of rank r. Find a $p\times q$ matrix A_1 of rank $s\leqq r$ such that $A_1=BDC$, where D is non-singular of order s, $A_2=A-A_1$ of rank $r-s$, B of order $p\times s$, C of order $s\times q$, and both B and C are of rank s. It is shown that all such A_1 are obtained by choosing an arbitrary X of order $s\times p$ and a Y of order $s\times q$ such that XAY' is non-singular and putting $D=(XAY')^{-1}$, $B=AY'$, $C=XA$. *O. Taussky-Todd* (Pasadena, Calif.).

Lense, Josef. Bemerkungen zur Verwendung der Elementarteiler bei Paaren von nicht zerfallenden Kegelschnitten. Math. Z. **67** (1957), 147–152.

Given two conic sections whose 3 by 3 matrices are A and B, the author enumerates the six categories with respect to the elementary divisors of $A+\lambda B$. He interprets geometrically the categories and finds canonical forms for each. *B. W. Jones* (Boulder, Colo.).

Samelson, Hans. On the Perron-Frobenius theorem. Michigan Math. J. **4** (1957), 57–59.

The fact that a positive $n\times n$ matrix T has exactly one positive eigenvector is very briefly proved as follows: The plane $\sum_{i=1}^n x_i=1$ intersects the first orthant in an $(n-1)$-simplex with interior S, and T induces a mapping $\tilde T$ of S on itself by associating the intersection of a ray from O with S with that of its image. This mapping shrinks the distances of the Hilbert geometry with range S. It follows that $\bigcap_1^\infty \tilde T^i(S)$ consists of exactly one point, which furnishes the eigenvector. *H. Busemann.*

Marcus, M.; and Moyls, B. N. On the maximum principle of Ky Fan. Canad. J. Math. **9** (1957), 313–320.

Let $A_1, \cdots, A_m$ be arbitrary n by n complex matrices. Let the square roots of the eigenvalues of the hermitian matrix $A_\sigma^*A_\sigma$ be denoted by $\alpha_{\sigma1}\geqq\alpha_{\sigma2}\geqq\cdots\geqq\alpha_{\sigma n}$. Let $U_1, \cdots, U_m$ be unitary matrices and $x_1, \cdots, x_k$ orthonormal vectors. Ky Fan has shown [Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 760–766; MR **13**, 661] that

$$\text{(1)}\qquad \max\left|\sum_{i=1}^k (U_1A_1\cdots U_mA_mx_i, x_i)\right|=\sum_{i=1}^k\left(\prod_{\sigma=1}^m \alpha_{\sigma i}\right),$$

$$\text{(2)}\qquad \max|\det\{(U_1A_1\cdots U_mA_mx_i, x_j)\}|=\prod_{l=1}^k\left(\prod_{\sigma=1}^m \alpha_{\sigma l}\right)\qquad (i, j=1, \cdots, k).$$

These maxima are taken with respect to all unitary matrices U_σ and sets of k orthonormal vectors $x_1, \cdots, x_k$.

Noting that the right hand side of (1) is the symmetric function $E_1(\prod\alpha_{\sigma1}, \cdots, \prod\alpha_{\sigma k})$ and that the right hand side of (2) is $E_k(\prod\alpha_{\sigma1}, \cdots, \prod\alpha_{\sigma k})$, the authors proceed to give a maximum principle for the general symmetric function. Denoting by $\phi(x_1, \cdots, x_k; U_1, \cdots, U_m)$ the absolute value of the sum of all $r\times r$ determinants of the form

$$\text{(3)}\qquad \det\{(U_1A_1\cdots U_mA_mx_{i_p}, x_{i_q})\}\qquad (p, q=1, \cdots, r;\ 1\leqq i_1<i_2<\cdots<i_r\leqq k),$$

they show that for $1\leqq r\leqq k\leqq n$

$$\text{(4)}\qquad \max\phi(x_1, \cdots, x_k; U_1, \cdots, U_m)=E_r\left(\prod_{\sigma=1}^m\alpha_{\sigma1}, \cdots, \prod_{\sigma=1}^m\alpha_{\sigma k}\right).$$

The maximum is again with respect to sets of k orthonormal vectors and m unitary matrices.

As an application the following theorem is proved. Let A and B be $n\times n$ matrices. Let the eigenvalues of A^*A, B^*B, and $\sigma B^*A^*AB+\delta ABB^*A^*$ arranged in non-increasing order be denoted by α_i^2, β_i^2, and φ_i^2, respectively. Here σ and δ are non-negative real numbers and $\sigma+\delta=1$. Then for $0\leqq s\leqq 1$, $1\leqq r\leqq k\leqq n$,

$$\text{(5)}\qquad E_r(\varphi_n^{2s}, \cdots, \varphi_{n-k+1}^{2s})\geqq(2\sigma^\sigma\delta^\delta)^{r(s-1)}\left(\prod_{j=1}^r \alpha_{n-j+1}^\sigma\beta_{n-j+1}^\delta\right)\times[E_r(\beta_n^{2s}, \cdots, \beta_{n-k+1}^{2s})]^\sigma[E_r(\alpha_n^{2s}, \cdots, \alpha_{n-k+1}^{2s})]^\delta.$$

{The results (1) and (2) of Ky Fan hold for completely continuous operators A_σ in a Hilbert space. The result (4) is proved in the notation of Grassman algebra, and the authors state that it is restricted to finite-dimensional spaces. However, if we deal with a Hilbert space, and if for some orthonormal vectors $x_1, \cdots, x_k$ and some unitary operators $U_1, \cdots, U_m$, $\phi(x_1, \cdots, x_k, U_1, \cdots, U_m)$ is greater than the right-hand side of (4), then the same is true for the finite-dimensional space spanned by the vectors x_i, A_mx_i, $U_mA_mx_i$, $\cdots$, $U_1A_1\cdots U_mA_mx_i$ $(i=1, \cdots, k)$. Thus, this situation cannot occur, and we have $\phi\leqq E_r$. That the equality is attained can be proved in the same way as for the finite-dimensional case. Thus, (4) holds for completely continuous operators in a Hilbert space.} *H. F. Weinberger* (Madison, Wis.).

Brenner, J. L. Bounds for determinants. II. Proc. Amer. Math. Soc. **8** (1957), 532–534.

[For part I see Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 452–454; MR **15**, 926.] Let A be an arbitrary n by n matrix with real or complex elements, a_{ij}, and let $m_i=\max_j |a_{ij}|$, $j\neq i$. The author proves the following inequality:

$$|\det A|\geqq\left[1-\sum_1^n m_j(|a_{jj}|+m_j)^{-1}\right]\prod_1^n (|a_{ii}|+m_i).$$

He also gives an upper bound for $|\det A|$ which he says may be derived by similar arguments. Both of these are suitable for use with automatic computing machinery.
B. W. Jones (Boulder, Colo.).

See also: Charles, p. 117; Murase, p. 119; Goldberg, p. 123; Auslander, p. 147.

Polynomials

Shapiro, Harold S. The range of an integer-valued polynomial. Amer. Math. Monthly **64** (1957), 424–425.

A proof of the following theorem. Let $P(x)$ and $Q(x)$ be polynomials which are integer-valued at the integers, of degrees p and q respectively. If $P(n)$ is of the form $Q(m)$ for all n, or even for infinitely many blocks of consecutive integers of length $\geqq p/q+2$, then there is a polynomial $R(x)$ such that $P(x)=Q[R(x)]$.
From the author's summary.

See also: Remmert, p. 170; Head, p. 177.

Partial Order, Lattices

Hartmanis, Juris. A note on the lattice of geometries. Proc. Amer. Math. Soc. **8** (1957), 560–562.

This is a continuation of a previous paper [same Proc. **7** (1956), 571–577; MR **18**, 6] in which the author defined the lattice of geometries on a set S. Here it is shown that this lattice is complemented with a special type of complement; furthermore, the lattice has only the trivial homomorphisms, and the group of automorphisms is the symmetric group on S. *O. Ore.*

Robison, G. B.; and Wolk, E. S. The imbedding operators on a partially ordered set. Proc. Amer. Math. Soc. **8** (1957), 551–559.

Some elementary properties of the lattice of imbedding operators on a partially ordered set are derived and sufficient conditions are given that this lattice be a chain.
R. P. Dilworth (Pasadena, Calif.).

Smith, Edgar C., Jr. A distributivity condition for Boolean algebras. Ann. of Math. (2) **64** (1956), 551–561.

The author considers a kind of transfinite distributivity for Boolean algebras, called property $(P_\mathfrak{n})$, which, by its definition, is weaker than the usual $\mathfrak{n}$-distributivity condition. He shows that it is properly weaker by exhibiting, for every regular cardinal $\mathfrak{n}$, an atomless complete Boolean algebra which has the property $(P_\mathfrak{n})$ but is not $\mathfrak{n}$-distributive. On the other hand, the property $(P_\mathfrak{n})$ does imply $\mathfrak{m}$-distributivity for every $\mathfrak{m}<\mathfrak{n}$. The new condition relates especially well to $\mathfrak{n}$-quotient algebras. First, every $\mathfrak{n}$-complete Boolean algebra having the property $(P_\mathfrak{n})$ is an $\mathfrak{n}$-quotient algebra, that is, is isomorphic to an $\mathfrak{n}$-complete field of sets modulo an $\mathfrak{n}$-ideal. In the converse direction, the author shows that for certain $\mathfrak{n}$, $\mathfrak{n}$-quotient algebras have the property $(P_\mathfrak{n})$, and that the validity of this assertion for all non-limiting infinite cardinal numbers $\mathfrak{n}$ is equivalent to the generalized continuum hypothesis. *L. H. Loomis* (Cambridge, Mass.).

Scott, Dana. The independence of certain distributive laws in Boolean algebras. Trans. Amer. Math. Soc. **84** (1957), 258–261.

Terminology and notations are taken from the paper reviewed above. The author gives an example of a complete Boolean algebra that is (β,γ)-distributive for every $\beta<\alpha$ and every γ, but not (α,α)-distributive, where α is a regular cardinal number. This strengthens a result of E. C. Smith [see the paper reviewed above]. To this end a 0-dimensional Hausdorff space X is constructed, the points of which are the subsets of α. If x and y are two subsets of X, then the interval $[x, y]$ is the set of all subsets z, such that $x\leqq z\leqq y$. A basis for the open sets of X is the collection of all intervals $[x, y]$ such that $x\cup(\alpha-y)\subset\beta$ for some $\beta<\alpha$. If N is the Boolean algebra of all regular open sets of X, then the author shows that N is (β,γ)-distributive for every $\beta<\alpha$ and every γ, but not (α,α)-distributive.
P. Dwinger (W. Lafayette, Ind.).

★**Korobkov, V. K. Realization of symmetric functions in the class of π (series-parallel) circuits.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1957. 6 pp.

Translated from Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 260–263 [MR **18**, 372].

See also: Behrens, p. 116; Popovici, p. 235.

Fields, Rings

Jaffard, Paul. Un exemple concernant les groupes de divisibilité. C. R. Acad. Sci. Paris **243** (1956), 1264–1266.

Die multiplikative Gruppe G eines Körpers, in dem eine endliche diskrete Hauptordnung ausgezeichnet ist, ist normal, d.h. als direktes Produkt zyklischer Gruppen darstellbar. Für einen Divisor d und endlich viele Bewertungen $B_1, \cdots, B_n$ existiert dabei in G stets ein Vielfaches x von d, das in $B_1, \cdots, B_n$ denselben Wert wie d hat. Verfasser gibt ein Beispiel einer normalen Gruppe, die diese Bedingung nicht erfüllt. *P. Lorenzen.*

Tiago de Oliveira, J. Residuals of systems and radicals of rings. Univ. Lisboa. Revista Fac. Ci. A. (2) **5** (1956), 177–248. (Portuguese. English summary)

The only results in this paper which concern the theory of rings and are not already in the literature are obvious consequences of the definitions. The author is chiefly interested in what he calls "primordial rings", i.e. rings, without nilpotent ideals, and in particular in prime rings; but he is apparently unaware of Johnson's work on the latter. The part of the paper concerned with rings is preceded by a first chapter on much more general algebraic systems, which does not go any deeper than the second part. *J. Dieudonné* (Evanston, Ill.).

Geddes, A. On the embedding theorems for complete local rings. Proc. London Math. Soc. (3) **6** (1956), 343–354.

L'auteur appelle anneau local faible (weak local ring) un anneau commutatif $\mathfrak{R}$ avec élément unité 1 dont les éléments non inversibles forment un idéal $\mathfrak{m}$ tel que $\bigcap_{n=1}^{\infty} \mathfrak{m}^n=(0)$. Dans J. London Math. Soc. **29** (1954), 334–341 [MR **16**, 213] le même auteur démontre le théorème de Cohen sur l'existence de corps de coefficients pour les anneaux locaux complets dans le cas d'égales caractéristiques. Ici on démontre pour ces anneaux la deuxième partie de ce théorème de Cohen, c'est à dire, l'existence d'un anneau de coéfficients (ou de Cohen) dans le cas d'inégales caractéristiques. Plus précisément, si la caractéristique du corps résiduel $\mathfrak{R}/\mathfrak{m}$ est p, alors $\mathfrak{R}$ contient un sous-anneau local complet C, d'idéal maximal engendré par $p1$, concordant avec $\mathfrak{R}$ et tel que l'homomorphisme de C dans $\mathfrak{R}/\mathfrak{m}$, induit par l'homomorphisme canonique de $\mathfrak{R}$ sur $\mathfrak{R}/\mathfrak{m}$, soit surjectif. La démonstration est tout a fait différente de celle que Nagata a donné aussi pour les anneaux locaux faibles [Nagoya Math. J. **1** (1950), 63–70; **5** (1953), 145–147; MR **13**, 7; **14**, 529] et me semble beaucoup plus simple. *E. Lluis.*

Batho, Edward H. Non-commutative semi-local and local rings. Duke Math. J. **24** (1957), 163–172.

Let R be a noncommutative ring with unit. The author calls R local (resp. semilocal) in case R satisfies the maximum condition on left ideals, R modulo its radical J is simple (resp. semisimple — but this is automatic) with minimum condition, and $\bigcap_{n=1}^{\infty} J^n=0$. If R is complete (in the usual J-adic topology), idempotents and matrix units can be raised. Hence every complete local ring is a matrix ring over a completely primary, complete local ring. Similarly, a complete semilocal ring is algebraically a group-direct sum $A_1\oplus A_2\oplus\cdots\oplus A_n\oplus N$, where each $A_i=e_iRe_i$ is a matrix ring over a completely primary ring, and $N\subset J$; but this is not a ring-direct sum unless ($N=0$ and) the e_i's are central idempotents; the author does not indicate when this can be expected. *D. Zelinsky.*

Algebras

Behrens, Ernst-August. Zweiseitige Ideale in Algebren endlichen Ranges. Math. Ann. **132** (1956), 95–105.

A semi-simple associative algebra $\mathfrak{o}$ of finite order over its ground field K is a direct sum of simple algebras $\mathfrak{m}_1, \cdots, \mathfrak{m}_s$, and its ideals are the zero ideal and the direct sums of any selection of the ideals $\mathfrak{m}_i$. The author asks what remains of this simple recipe for ideals if we drop the assumptions of semi-simplicity and associativity, and he answers the question for algebras satisfying assumption (A) below.

A submodule $\mathfrak{a}$ of the non-associative algebra $\mathfrak{o}$ can be expressed as $\mathfrak{a}=\mathfrak{o}E$, where E is an idempotent in $\{K\}_{n\times n}$, the ring of all linear transformations of $\mathfrak{o}$; and then $\mathfrak{a}$ is an ideal of $\mathfrak{o}$ if and only if $ETE=ET$ for all T of $T(\mathfrak{o})$, the transformation ring of $\mathfrak{o}$, i.e., the subalgebra of $\{K\}_{n\times n}$ generated by the identity and all left and right multiplications of $\mathfrak{o}$ [Albert, Ann. of Math. (2) **43** (1942), 685–707; MR **4**, 186]. E may not be in $T(\mathfrak{o})$.Assumption (A) is: There is a maximal ideal chain $0\prec\mathfrak{a}_1\prec\mathfrak{a}_2\prec\cdots\prec\mathfrak{a}_s=\mathfrak{o}$, such that $\mathfrak{a}_i=\mathfrak{o}E_i$ ($i=1, \cdots, s$), where the idempotents E_i are all in $T(\mathfrak{o})$. Then we can define submodules $\mathfrak{m}_1, \cdots, \mathfrak{m}_s$ of $\mathfrak{o}$ such that $\mathfrak{a}_r=\mathfrak{m}_1+\cdots+\mathfrak{m}_r$ ($r=1, \cdots, s$), (module-theoretic direct sums), and it is shown that the ideals of $\mathfrak{o}$ are the zero ideal and the (ring-theoretic) direct sums of certain selections of the $\mathfrak{m}_i$. It is also shown that (A) implies a lattice-isomorphism between the lattice of ideals in $\mathfrak{o}$ and the lattice of certain associated ideals in T($\mathfrak{o}$).

The author promises a further investigation of the algebras which fulfil (A) [see the review below], and meanwhile shows that (A) is satisfied if (as in the associative semi-simple case) $\mathfrak{o}$ has a maximal chain of ideals none of whose difference algebras is a zero algebra. But this condition is not necessary for (A) even when $\mathfrak{o}$ is associative. *I. M. H. Etherington* (Edinburgh).

Behrens, Ernst-August. Algebren mit vorgegebenem endlichen, distributiven Idealverband. Math. Ann. **133** (1957), 79–90.

This paper continues the author's study of non-associative algebras $\mathfrak{o}$ of finite dimension over a field which satisfy the condition (*): $\mathfrak{o}$ has a maximal chain of two-sided ideals $0=\mathfrak{a}_0\prec\mathfrak{a}_1\prec\cdots\prec\mathfrak{a}_s=\mathfrak{o}$ such that $\mathfrak{a}_i=\mathfrak{o}E_i$ with E_i an idempotent of the multiplication ring of $\mathfrak{o}$ ($i=1, \cdots, s$). The two-sided ideals of the algebras satisfying (*) were determined in the paper reviewed above. He now uses this result to show that the lattice $\mathfrak{L}(\mathfrak{o})$ of two-sided ideals of an algebra $\mathfrak{o}$ which satisfies (*) is finite and distributive, and he determines the join-irreducible elements of $\mathfrak{L}(\mathfrak{o})$. Employing an explicit matric representation of the join-irreducible elements of a finite distributive lattice $\mathfrak{L}$, he then shows that every such lattice $\mathfrak{L}$ has the form $\mathfrak{L}(\mathfrak{o})$ for a suitable algebra $\mathfrak{o}$ which satisfies (*). The paper concludes with similar arguments which establish that, for a non-associative algebra $\mathfrak{o}$ with identity element, $\mathfrak{L}(\mathfrak{o})$ is a Boolean algebra if and only if one of the following equivalent conditions holds: (1) $\mathfrak{o}$ has a maximal chain of two-sided ideals none of whose factor algebras is a zero algebra, (2) $\mathfrak{o}$ is a direct sum of simple algebras each with identity element, and (3) every two-sided ideal of $\mathfrak{o}$ is idempotent. *M. F. Smiley.*

See also: Tiago de Oliveira, p. 115; Eilenberg and Nakayama, p. 118; Pontrjagin, p. 152.

Groups and Generalizations

Solian, Alexandru. Über die n-Vollständigkeit in Gruppen. Rev. Math. Pures Appl. **1** (1956), no. 1, 5–22.

A translation from the Romanian of the article reviewed in MR **17**, 1183.

Gerstenhaber, Murray. On canonical constructions. III. Proc. Amer. Math. Soc. **7** (1956), 543–550.

[For Part II see Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 881–883; MR **18**, 870.] In the first paper of this series [Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 233–236; MR **17**, 581] the author illustrated his general theory of canonical constructions by proving that every automorphism of the symmetric group Σ_n of permutations on n elements is an inner automorphism, except when $n=6$. The case of infinite n is the interesting one, the finite case having been settled by J. König and O. Hölder.

In this paper the author defines the alternating group A_n to be the group generated by all 3-cycles of Σ_n. He then proves that if G is any subgroup of Σ_n containing A_n, then any automorphism of G may be extended uniquely to an inner automorphism of Σ_n, except when $n=3$ or 6

(when $n=3$ the extension is not unique, when $n=6$ it is impossible). Again the case where n is infinite is the interesting one. The proof is quite long, and depends on an abstract characterization of the elements of A_n which are 3-cycles.

In an appendix, the author fills in a gap in the original paper, making use of a characterization of transpositions due to W. R. Scott as follows: The element x of Σ_n is a transposition if and only if (1) the order of x is two; (2) there exists an element y of order two such that the order of $xy^{-1}xy$ is three; (3) for no element y of order two does the order of $xy^{-1}xy$ exceed three. *O. Frink.*

Pic, Gh. De la caractérisation des groupes cycliques. Com. Acad. R. P. Romîne 6 (1956), 235–238. (Romanian. Russian and French summaries)

F. Szász [Rev. Math. Pures Appl. **1** (1956), no. 2, 13–16; MR **18**, 789] showed that a group G is cyclic if and only if it has "property P", namely, that every cyclic subgroup of G is for some k the group generated by the kth powers of elements of G. The author investigates a parallel but weaker "property Q", namely, that every cyclic subgroup of G is for some k the group generated by the kth roots in G of the unit element. It is shown that Q implies the following properties: all subgroups and quotient groups of G also have property Q; G is abelian; G has at most one subgroup of a given order; G is cyclic if and only if it is finite. Finally, it is shown that Q is equivalent to the property that G is isomorphic to a subgroup of the additive group of rational numbers mod 1.

I. M. H. Etherington (Edinburgh).

Charles, B. Suites décroissantes d'espaces vectoriels. Ann. Univ. Sarav. **5** (1956), 107–111 (1957).

In proving Ulm's Theorem, one must consider a transfinite descending sequence of subspaces P_α of a vector space P over GF(p), with $\bigcap P_\alpha=0$ (for Ulm's Theorem, P_α is the set of elements of order p and height $\geq\alpha$), and one asks if $P_\alpha=Q_\alpha\oplus P_{\alpha+1}$ with $\sum Q_\alpha=P$. The author gives an incorrect proof that the answer is yes when P has a denumerable basis. *D. Zelinsky* (Evanston, Ill.).

Struik, Ruth Rebekka. Notes on a paper by Sanov. Proc. Amer. Math. Soc. **8** (1957), 638–641.

This paper gives direct proofs of some formulae on higher commutators, originally derived by Sanov [Izv. Akad. Nauk SSSR. Ser. Mat. **15** (1951), 477–502; MR **14**, 722] who used the Baker-Hausdorff formula in his proof. Let F_i be the ith term of the lower central series of a free group F generated by two elements u and v, and let $F(k)$ be the fully invariant subgroup generated by kth powers of elements of F. Then it is shown that

$$(u, v)^n \in F(2n)F_3 \text{ and } (u, v, v)^n \in F(3n)F_4.$$

Marshall Hall, Jr.

Higman, Graham. Finite groups having isomorphic images in every finite group of which they are homomorphic images. Quart. J. Math. Oxford. Ser. (2) 6 (1955), 250–254.

It is proved that a finite group G possesses the property of the title if and only if it is either a cyclic group or the direct product of two cyclic groups of which one has square-free order and the order of at least one factor is not twice an odd number. In addition, it is proved that for any normal divisor R of a free group F, the factor group $F/[R, R]$ has no torsion.

B. I. Plotkin. (RŽMat 1957, No. 137).

Schiek, Helmut. Über die Darstellungen der Gruppen mit quadratfreier Ordnungszahl. Math. Nachr. **14** (1955), 287–307 (1956).

A group G with square-free order g has been shown by Hölder to have a maximal cyclic normal subgroup $\{T\}$ of order $n=g/m$ and to be generated by S and T, where $S^m=T^n=E$, $TS=ST^a$, where a has exponent m (mod n). The author, in this extract from his thesis [Leipzig, 1942], shows that all the irreducible representations of such a group are equivalent to monomial representations of factor groups G/N. They are in one to one correspondence with the classes of elements of G. If $m=\prod p_i$, and $n=\prod q_j$, then a has the form $a\equiv\prod a_i$ (mod n), where a_i belongs to the exponent p_i (mod n). Hölder has called a prime factor q_j of n conjugate or not to p_i according as a_i has exponent p_i or 1 (mod q_j). To any factor $\mu=\prod p_\gamma$ of m corresponds a maximal factor $n_\mu=\prod q_\tau$ such that no q_τ is conjugate to any factor p_γ of μ. In particular

$$n_m=(a-1, n).$$

If $(h, m)=m/\mu$, and $(i, n_\mu)=n_\mu/\nu$ then S^hT^i has order $\mu\nu$; and conversely there are elements of any order $\mu\nu$ such that $\mu|m$ and $\nu|n_\mu$. The commutator group K of G is a cyclic group generated by T^{a-1}, whose order is shown to be n/n_m. If $\mu|m$ and $\nu|n$, the commutator group of a subgroup U of order $\mu\nu$ is of order ν/ν_μ. A subgroup U of order $\mu\nu$ can be generated by $S'=S^{m/\mu}T^{x\nu_\mu}$ and $T'=T^{n/\nu}$, and is normal in G if and only if n/ν divides n_μ. In the latter case there is a unique subgroup of order $\mu\nu$. The author examines and describes a complete set of non-equivalent monomial representations of G, generated by pairs of normal subgroups whose relative factor groups are cyclic, and sets them in one to one correspondence with the classes of G. *J. S. Frame* (East Lansing, Mich.).

★**Mackey, G. W. The theory of group representations.** Lecture notes (Summer, 1955) prepared by Dr. Fell and Dr. Lowdenslager. Dept. of Math., Univ. of Chicago, Chicago, Ill. Three volumes. Vol. 1, pp. 1–57; vol. 2, pp. 58–111; vol. 3, pp. 112–182.

Adyan, S. I. On the problem of divisibility in semigroups. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 747–750. (Russian)

This paper considers semigroups with cancellation which are given by a finite system of generators and a finite number of defining relations. The problem of (right) divisibility for the semigroup $\mathfrak{A}$ consists in indicating an algorithm which tells us, for arbitrary words Q and R in $\mathfrak{A}$, whether there exists in $\mathfrak{A}$ a word X such that $XQ=R$. In the article, a semigroup is constructed for which the problem of divisibility is insoluble.

The problem of existence of an inverse element, for the semigroup $\mathfrak{A}$ with identity, consists in indicating an algorithm which tells us whether there exists, for any given word $A \in \mathfrak{A}$, a (right) inverse element. It turns out that it is impossible to construct a semigroup with insoluble problem of existence of an inverse element. Such a semigroup may be constructed only in the case of a countable number of generators and defining relations. On the other hand, an algorithm which solves the problem of existence of an inverse element in every semigroup (with a finite number of generators and defining relations) is impossible.

A. S. Esenin-Vol'pin (RŽMat 1956, no. 7169).

Yamada, Miyuki. Compositions of semigroups. Kōdai Math. Sem. Rep. **8** (1956), 107–111.

Yamada, Miyuki. Correction to compositions of semigroups. Kōdai Math. Sem. Rep. **8** (1956), 189.

If a semigroup has a two-sided unit but no other idempotent, we shall call such a semigroup a hypogroup. Let L be a given semilattice and S_δ be, for each $\delta \in L$, a given hypogroup. Let S be the direct sum, i.e., the disjoint sum of all S_δ: $S=\sum_{\delta\in L} S_\delta$.

The author solves the problem of constructing every possible semigroup $S(\circ)$ which consists of all elements of S and in which a product $\circ$ is defined in such a way that (1) for each $\delta \in L$, S_δ is a subsemigroup of $S(\circ)$, i.e. $a_\delta \circ b_\delta = a_\delta b_\delta$ for each a_δ, $b_\delta \in S_\delta$, (2) for each $\alpha, \beta \in L$, $S_\alpha \circ S_\beta \subset S_{\alpha\beta}$. Further, an analogous problem for the commutative case is solved in the present paper. For the case when the semigroups S_δ are not hypogroups the author gives an example where no semigroup $S(\circ)$ exists for which (1), (2) are true. *F. Šik* (Brno).

Schieferdecker, Eberhard. Fastperiodische Fortsetzung von Funktionen auf Halbgruppen. Math. Nachr. **14** (1955), 253–261 (1956).

Let $\mathfrak{H}$ be a semigroup that can be imbedded in a group, and let $\mathfrak{Q}^*(\mathfrak{H})$ be the group containing $\mathfrak{H}$ constructed by J. Lambek [Canad. J. Math. **3** (1951), 34–43; MR **12**, 481] in this case. Let $\mathfrak{H}$ be a semigroup with unit and let $\mathfrak{H}_0 \subset \mathfrak{H}$ be a set of generators for $\mathfrak{H}$. Let f_0 be a complex function on $\mathfrak{H}_0$. Then f_0 can be extended to an almost periodic function on $\mathfrak{Q}^*(\mathfrak{H})$ if and only if f_0 can be extended to a function f defined on $\mathfrak{H}$ that is almost periodic in the sense of W. Maak [Acta Math. **87** (1952), 33–58; MR **13**, 910]. For every such f_0 the almost periodic extension over $\mathfrak{Q}^*(\mathfrak{H})$ is unique. The author has given analogous theorems in another paper [Arch. Math. **6** (1955), 428–438; MR **17**, 346]. *E. Hewitt.*

See also: Karrer, p. 114; Jaffard, p. 115; Taylor, p. 118; Goldberg, p. 123; Taussky and Todd, p. 123; Altmann, p. 133; Leech, p. 165; Atiyah, p. 172.

Homological Algebra

Taylor, Robert L. Compound group extensions. III. Trans. Amer. Math. Soc. **79**, 490–520 (1955).

In this paper the author continues the work of parts I–II [same Trans. **75** (1953), 106–135, 309–310; MR **15**, 599]. The results obtained are quite technical in nature, and difficult to state without introducing a good deal of notation. *J. C. Moore* (Princeton, N.J.).

Cobbe, Anne P.; and Taylor, R. L. On Q-kernels with operators. Quart. J. Math. Oxford Ser. (2) **8** (1957), 13–38.

A Q-kernel is a pair of groups Q and K together with a homomorphism θ: $Q\to A(K)/I(K)$. Here A and I denote the (additive!) groups of automorphisms resp. inner automorphisms of K. An extension of the kernel is an exact sequence $0\to K\to E\to Q\to 0$ suitably inducing θ. Take Z=center K. It is known [Eilenberg and MacLane, Ann. of Math. (2) **48** (1947), 326–341; MR **9**, 7] that the set EXT of equivalence classes of extensions is a "coset" of the cohomology group $H^2(Q; Z)$; i.e., H^2 operates in a simply transitive fashion on EXT. The present paper analyses the coset in the case of kernels with operator in a group P; i.e., P operates on both Q and K, and

$$\theta(pq)=\nu(p)+\theta(q)-\nu(p),$$

where $\nu(p)$ denotes the automorphism class of the operation of p on K. This assumption yields operators of P on EXT which have the property $p(g+e)=pg+pe$ for $p \in P$, $g \in H^2$, $e \in$ EXT, where $g+e$ is the result of operating on e with g. Now any set with such operators from the groups P and H^2, simply transitive under the operators of H^2, corresponds naturally (Theorem 1) to an element of $H^1(P, H^2)$; therefore the EXT of each extensible kernel with operators P yields an element χ of $H^1(P, H^2(Q, Z))$ This invariant χ can be neatly calculated from the vector-cohomology obstruction χ_W of the kernel with operators [J. H. C. Whitehead, Quart. J. Math. Oxford Ser. (2) **1** (1950), 219–228; MR **12**, 156], using the fact that χ_W determines the Eilenberg-MacLane obstruction invariant, which latter vanishes when the kernel has any extensions. This yields (Theorem 3) a characterization of those χ which can arise from some kernel with operators. This result is refined in the special case when P operates trivially on K. Set $\bar{Q}=Q/Q_0$, Q_0 the subgroup generated by all $(pq)q^{-1}$, for $p \in P$, $q \in Q$, and set

$$H_0{}^3(\bar{Q}, Z)=\ker\{H^3(\bar{Q}, Z)\to H^3(Q, Z)\}.$$

Each nonhomogeneous cocycle f in a cohomology class of $H_0{}^3$ can therefore be written as δg for some g in $C^2(Q, Z)$. The function $g(p^{-1}x, p^{-1}y)-g(x, y)$ for $x, y \in Q$ yields a (vector) cocycle in $Z^1(P, H^2(Q, Z))$; the passage from f to g induces a homomorphism γ: $H_0{}^3(\bar{Q}; Z)\to H^1(P; H^2(Q,Z))$. Theorem 5 asserts that the image of γ consists precisely of those classes χ realized by the EXT of some kernel with operators. This corrects an earlier analysis [A. P. Cobbe, ibid. **2** (1951), 269–285; MR **13**, 529], as shown by an example where P and Q are both cyclic. The calculations for this example employ the usual resolutions for cyclic groups; they therefore replace the above definition of γ, which depends on the functorial character of the standard resolution, by an ingenious definition of γ using arbitrary projective resolutions. *S. MacLane.*

Eilenberg, Samuel; and Nakayama, Tadasi. On the dimension of modules and algebras. V. Dimension of residue rings. Nagoya Math. J. **11** (1957), 9–12.

[For parts I–IV see same J. **8** (1955), 49–57; **9** (1955), 1–16, 67–77; **10** (1956), 87–95; MR **16**, 993; **17**, 453, 579; **18**, 9.] The authors prove the following two theorems for a semi-primary ring Λ with radical N. a) If a is a two-sided ideal in Λ such that $a\subset N^2$ and gl dim $(\Lambda/a)\leqq 1$, then $a=0$. b) If r is a right ideal in Λ such that $r\, N\subset Nr\subset N^2$ and gl dim $(\Lambda/Nr)\leqq n$, $n>1$, then $Nr^{n-1}N=0$. As a consequence of these results, they show that if gl dim $(\Lambda/N^k)\leqq n$, $k>1$, $n>1$, then $N^{(n-1)(k-1)+2}=0$. If gl dim $(\Lambda/N^2)\leqq n$, $n\geqq 0$, then $N^{n+1}=0$.

The proofs of a) and b) above lean heavily on an orthogonality relation $a \perp A$, where a is a subset of Λ and A is a left Λ-module. It is defined by the condition $aP=0$, where φ: $P\to A$ is a minimal epimorphism (P projective) [see Eilenberg, Ann. of Math. (2) **64** (1956), 328–336; MR **18**, 558]. *D. Buchsbaum.*

★ **Serre, Jean-Pierre. Sur la dimension homologique des anneaux et des modules noethériens.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 175–189. Science Council of Japan, Tokyo, 1956.

The author gives an exposition of the results of M. Auslander and the reviewer [Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 36–38; MR **17**, 705] and completes these results, notably by giving a homological characterization of regular local rings.

All the rings A considered in the paper are commutative noetherian rings with identity element, and all A-modules E are assumed finitely generated and unitary. If A is a local ring with unique maximal ideal $\mathfrak{m}$, and E is an A-module, a sequence of elements $a_1, \cdots, a_q$ in $\mathfrak{m}$ is called an E-sequence if for each $i=1, \cdots, q$, a_i is not a zero divisor for $E/(a_1, \cdots, a_{i-1})E$. It is shown that every E-sequence can be extended to a maximal E-sequence, and if $a_1, \cdots, a_q$ is an E-sequence, then

$$\mathrm{dh}_A(E/(a_1, \cdots, a_q)E) = \mathrm{dh}_A E + q,$$

where $\mathrm{dh}_A(E)$ is the projective (or homological) dimension of E over A [H. Cartan and S. Eilenberg, Homological algebra, Princeton, 1956; MR **17**, 1040; here $\mathrm{dh}_A E$ is written $\dim_A E$]. Moreover, if the global dimension of A is $s<\infty$, then every maximal E-sequence has length equal to $s-\mathrm{dh}_A E$; and if A is a regular local ring of dimension n, then the global dimension of A is equal to n.

If A is an arbitrary local ring, and E an A-module, the author shows that any two maximal E-sequences have the same length. This length is called the codimension of E and denoted by $\mathrm{codh}_A E$. The utility of this concept is illustrated by applications to coherent algebraic sheaves, fractional divisorial ideals, and unique factorization.

The main original result of this paper is that a local ring of finite global dimension is regular. The important tool here is the construction of a free resolution of the residue field k of A which contains in it the exterior algebra complex (loc. cit., Chapter VIII) generated over A by a minimal generating set of $\mathfrak{m}$. Using this resolution, the author shows that the (linear) dimension of $\mathrm{Tor}_p^A(k, k)$ is greater than or equal to $\binom{n}{p}$ where $n=\dim_k(\mathfrak{m}/\mathfrak{m}^2)$ (linear dimension) [for a better estimate of this dimension see the paper reviewed below]. As a result of this characterization of regular local rings, the author proves that the ring of quotients $A_\mathfrak{p}$ of a regular local ring A with respect to a prime ideal $\mathfrak{p}$ is again regular. Other results, such as the Cohen-Macaulay theorem for regular local rings, are also proved here.

D. Buchsbaum.

Tate, John. Homology of Noetherian rings and local rings. Illinois J. Math. **1** (1957), 14–27.

In this paper the author makes systematic use of skew-commutative differential graded algebras over a commutative noetherian ring R to study the structure of

$$\mathrm{Tor}^R(R/M, R/N).$$

(Such an algebra is called an R-algebra.) He does this by first showing that there always exists a free resolution X of the residue class ring R/M which is an R-algebra. The algebra structure of $\mathrm{Tor}^R(R/M, R/N)$ can then be determined directly from these R-algebra resolutions of R/M and R/N.

To prove that R/M always has a free R-algebra resolution, the author introduces the device of killing cycles in an arbitrary R-algebra. Specifically, he shows that if X is an R-algebra, and t is a $(\rho-1)$-dimensional cycle $(\rho>0)$, then there is a canonical way of constructing an R-algebra Y containing X such that $Y_\lambda = X_\lambda$ for $\lambda<\rho$ and

$$B_{\rho-1}(Y) = B_{\rho-1}(X) + Rt$$

(where $B_{\rho-1}(X)$ means the boundaries of the R-algebra X of dimension $\rho-1$). The procedure depends on the parity of ρ. If ρ is odd, Y is essentially the exterior algebra over X generated by an element T (of degree ρ) and $dT=t$. If ρ is even, Y is the twisted polynomial ring in one generator T over X, with $dT=t$. The algebra Y is denoted by the symbols $X\langle T\rangle$, $dT=t$.

The map $i_*: H(X)\to H(Y)$ induced by $i: X\to Y$ is shown to be a surjection (epimorphism) if the homology class τ of t is a skew non-zero divisor, i.e., if, for $\xi\in H(X)$, $\tau\xi=0$ implies $\xi=0$ if ρ is odd and $\xi\in\tau H(X)$ if ρ is even.

Using these methods, a special resolution is obtained which yields an efficient method for computing the homology and cohomology groups of a finitely generated abelian group.

Generalizations of results of Eilenberg (unpublished) and of Serre [see the paper reviewed above] are obtained. In particular, if R is a local ring and K is the residue field of R, denote by $B_q(R)$ the dimension of the vector space $\mathrm{Tor}_q^R(K, K)$ over K. Now, if R is not a regular local ring, then $B_r(R) \geqq \binom{n}{r} + \binom{n}{r-2} + \cdots$ and hence $\geqq 2^{n-1}$ for $r\geqq n$, where n is the minimum number of elements required to generate the maximal ideal of R. Therefore, one obtains a new proof of the fact that regular local rings are precisely those of finite global dimension (Serre). Moreover, if $B_r(R) = \binom{n}{r}$ for one single dimension $r\geqq 2$, then R is regular. This generalizes the result of Eilenberg, which was proved only for $r=2$ or 3.

D. Buchsbaum (Providence, R.I.).

THEORY OF NUMBERS

General Theory of Numbers

Murase, Itiro. Semimagic squares and non-semisimple algebras. Amer. Math. Monthly **64** (1957), 168–173.

This note is a continuation of L. M. Weiner's "The algebra of semi-magic squares" [same Monthly, **62** (1955), 237–239; MR **16**, 894]. Weiner showed that the set of all semi-magic matrices of order n forms a subalgebra of the matrix algebra of order n, and that it is a direct sum of two ideals, one being the set of matrices whose row and column sums are zero. Murase shows that this ideal is isomorphic to the matrix algebra of degree $n-1$ (provided n is not a multiple of the characteristic of the ground field). He further obtains what might be called demi-semi-magic squares, and shows that they form non-semi-simple algebras.

H. A. Thurston (Bristol).

★ **Delesalle, A. Carrés magiques.** Gauthier-Villars, Paris, 1956. iv+70 pp. 800 francs.

Part 1: Methods of constructing ordinary magic squares of all orders $n>2$ which yield for prime n,

$$(n-4)(n-3)n!^2 + 2(n-3)n!(n-1)! + 2(n-1)!^2$$

distinct squares. Part 2: Extension of the same methods

to construction of magic squares whose entries are not the integers 1, 2, $\cdots$, n^2. Part 3: Some sufficient conditions on n^2 numbers which insure that these can be used as entries in a magic square. *L. Moser.*

Carlitz, L. A note on Kummer's congruences. Arch. Math. 7 (1957), 441–445.

Let $\{a_m\}$ be a sequence of rational numbers that are integral with respect to a fixed prime p and assume that the Kummer congruence

$$\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}a_p{}^{r-s}a_{m+s(p-1)}\equiv 0 \pmod{p^r}$$

is satisfied for $m\geq r$. Let the sequence $\{b_m\}$ satisfy a similar condition and form the Hurwitz product $\{c_m\}$ of the sequences $\{a_m\}$, $\{b_m\}$ by means of $c_m=\sum_{s=0}^{m}\binom{m}{s}a_sb_{m-s}$. The author's principal objective is to investigate the circumstances under which

$$(*)\qquad \sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}k^{r-s}c_{m+s(p-1)}\equiv 0 \pmod{p^r}$$

for $m\geq r$ and some k. Three theorems are proved. Theorem 1 asserts that if $a_p\equiv b_p \pmod p$, then (*) is satisfied with $k\equiv a_p \pmod p$. Theorems 2 and 3 state corresponding results under slightly modified conditions for the sequences $\{a_m\}$ and $\{b_m\}$. *A. L. Whiteman.*

Selmer, Ernst S. The rational solutions of the Diophantine equation $\eta^2=\xi^3-D$ for $|D|\leq 100$. Math. Scand. 4 (1956), 281–286.

The author lists the number of generators (in the sense of Poincaré-Mordell-Weil) of infinite order for the rational solutions of the title equation for integers D in $|D|\leq 100$. He gives the appropriate number of linearly independent solutions. He states that he has not verified that they are in fact a system of generators for the rational solutions but that he has verified that no solution listed, or the sum or difference of solutions listed, is either 2 or 3 times a rational solution in the sense of addition on the curve. It is further stated whether the solutions given are $\sqrt{(-3)}$ times a solution of $\eta^2=\xi^3+27D$ in the sense of complex multiplication. Earlier tables covering a more restricted range of D have been given by the reviewer [Acta Math. **82** (1950), 243–273; MR **12**, 11] and by Podsypanin [Mat. Sb. N.S. **24** (66) (1949), 391–403; MR **11**, 81]. A list of errata to Podsypanin's table is given.

The author also lists for $50\leq D\leq 100$ elements of the purely cubic fields which are ideal squares but not the product of a unit by a square of a number of the field. This extends the table of the reviewer (loc. cit.) for $0<D\leq 50$ and supplements the author's table which gives other information about these fields [Avh. Norske Vid. Akad. Oslo. I. **1955**, no. 5; MR **18**, 286].

J. W. S. Cassels (Cambridge, England).

Watson, G. L. Least solutions of homogeneous quadratic equations. Proc. Cambridge Philos. Soc. **53** (1957), 541–543.

Let $f(x_1,\cdots,x_n)=\sum f_{ij}x_ix_j$ be an indefinite quadratic form with integral coefficients. Cassels [same Proc. **51** (1955), 262–264; **52** (1956), 602; MR **16**, 1002; **18**, 380] proved that if the equation $f(x_1,\cdots,x_n)=0$ has a nonzero integral solution, then it has one satisfying

$$\max|x_i|\leq k_n(\max|f_{ij}|)^{(n-1)/2},$$

where k_n depends only on n. It is here shown that the exponent $\frac{1}{2}(n-1)$ can be replaced by $\max(2, \frac{1}{2}r, \frac{1}{2}s)$, where r is the number of positive squares and s the number of negative squares when f is expressed as a sum of squares (with signs) of real linear forms. Further, if for convenience we suppose $r\geq s$, the exponent cannot be replaced by any number less than $\frac{1}{2}[rs^{-1}]$; this is an extension of a result of Kneser given by Cassels. Further negative results when $n=4$ or 5 are obtained by special constructions. *H. Davenport* (London).

Din, Deota. On Prouhet Lehmer problem. J. Sci. Res. Banaras Hindu Univ. **6** (1955–56), 221–226.

Denote, as usual, by $P(k, s)$ the least value of j such that the $k(s-1)$ diophantine equations

$$\sum_{i=1}^{j}x_{1i}{}^h=\sum_{i=1}^{j}x_{2i}{}^h=\cdots=\sum_{i=1}^{j}x_{si}{}^h\quad(0\leq h\leq k)$$

have a non-trivial solution in integers. Improving on recent results of Prakash Srivastava [same J. **5** (1955), no. 2, 59–62; MR **17**, 586] and S. Sastry the author shows that $P(4, s)\leq 2s+1$ and $P(6, s)\leq 3s+2$. Numerical examples show that $P(4, 3)\leq 7$, $P(4, 6)\leq 13$, $P(6, 3)\leq 11$, $P(6, 6)\leq 20$. The method is based on a device of Sastry [ibid. **1** (1950–1951), 1–4; MR **13**, 535]. *D. H. Lehmer.*

Aigner, Alexander. Die Unmöglichkeit von $x^6+y^6=z^6$ und $x^9+y^9=z^9$ in quadratischen Körpern. Monatsh. Math. **61** (1957), 147–150.

Using techniques of Fogels [Comment. Math. Helv. **10** (1938), 263–269] as developed by himself [Monatsh. Math. **56** (1952), 240–252; MR **14**, 452], the author shows that the title equations have no solutions in any field quadratic over the rationals. *J. W. S. Cassels.*

Wirsing, Eduard. Über die Dichte multiplikativer Basen. Arch. Math. **8** (1957), 11–15.

A sequence of integers $1\leq a_1<a_2<\cdots$ is called a multiplicative basis of order k if every integer is the product of k or fewer a's. Denote by $A(x)$ the number of a's not exceeding x. The author proves that for every k

$$\liminf_{x\to\infty} x^{-1}A(x)\log x>1,$$

and that for every ε and every k there exists a multiplicative base of order k satisfing

$$\liminf_{x\to\infty} x^{-1}A(x)\log x<1+\varepsilon.$$

The proof of the first result is easy, but of the second more difficult.

Previously Raikov proved [Mat. Sb. N.S. **3(45)** (1938), 569–576] that if $\{a_i\}$ is a basis of order k then

$$\limsup_{x\to\infty} x^{-1}A(x)(\log x)^{1-1/k}\geq\Gamma(1/k).$$

P. Erdös (Toronto, Ont.).

Barnes, E. S. On a theorem of Voronoi. Proc. Cambridge Philos. Soc. **53** (1957), 537–539.

Elementary algebraic proof of the fundamental theorem of Voronoi [J. Reine Angew. Math. **133** (1907), 97–178] that a positive definite quadratic form is extreme if, and only if, it is perfect and eutactic. *J. A. Todd.*

See also: Gloden, p. 181.

Analytic Theory of Numbers

Lehmer, D. H. On the roots of the Riemann zeta-function. Acta Math. **95** (1956), 291–298.
Lehmer, D. H. Extended computation of the Riemann zeta-function. Mathematika **3** (1956), 102–108.

These two papers describe numerical calculations which establish the result that the first 25,000 zeros of $\zeta(s)$ lie on $\sigma=\frac{1}{2}$, thereby verifying the Riemann hypothesis for $|t|\leqq 21{,}943$. Earlier computations by Titchmarsh [Proc. Roy. Soc. London. Ser. A. **151** (1935), 234–255; **157** (1936), 261–263] have established a corresponding result for $|t|\leqq 1468$, while Turing [Proc. London Math. Soc. (3) 3 (1953), 99–117; MR **14**, 1126] carried the result to 1540. Some flaws in the argument are noted below.

The analytic basis for the computation may be described as follows. Let

$$\vartheta(t)=-\frac{1}{2}t\log\pi+\Im\log\Gamma\left(\frac{1}{4}+\frac{1}{2}it\right)$$
$$=\frac{1}{2}t\log\frac{t}{2\pi}-\frac{t}{2}-\frac{\pi}{8}+O\left(\frac{1}{t}\right),$$
$$F(t)=e^{i\vartheta(t)}\zeta(\tfrac{1}{2}+it).$$

Then $F(t)$ is real for real t. We let $N(t)$ be the number of zeros ρ of $\zeta(s)$ such that $0\leqq\Re\rho\leqq 1$ and $0\leqq\Im\rho\leqq t$. As shown by Titchmarsh, if k is the integer nearest to $\vartheta(t)/\pi$ and if $\Re\zeta(\sigma+it)\neq 0$ for $\frac{1}{2}\leqq\sigma\leqq 2$, then $N(t)=k+1$. Further, we define the Gram point τ_n for real $n\geqq -9/8$ as the unique solution of $\tau\log\tau-\tau=n+\frac{1}{8}$ satisfying $\tau\geqq 1$. As $n\to\infty$, one then has $\tau_{n+1}-\tau_n\sim 1/\log n$ and $\tau_n\sim n/\log n$. Gram's law, which is not strictly true, asserts that $\zeta(\frac{1}{2}+it)$, and hence $F(t)$, has exactly one zero between $2\pi\tau_n$ and $2\pi\tau_{n+1}$. If $\Re\zeta(\sigma+2\pi i\tau_n)\neq 0$, $\frac{1}{2}\leqq\sigma\leqq 2$, then $N(2\pi\tau_n)=n+1$. One then examines the signs of $F(2\pi\tau_k)$ for $k=-1, 0, \cdots, n$. If these alternate, one is then assured of at least $n+1$ zeros for $\zeta(s)$ up to $2\pi\tau_n$ and these all lie on $\sigma=\frac{1}{2}$; the equation $N(2\pi\tau_n)=n+1$ now establishes that $\zeta(s)$ has no other zeros up to $2\pi\tau_n$ but those on $\sigma=\frac{1}{2}$. This, in principle, is the basis for all such calculations. In practice, a number of difficulties have to be surmounted. First, an efficient method for determining the sign of $F(t)$ must be devised. Second, Gram's law does not hold universally, so that the numbers $F(2\pi\tau_k)$ do not always alternate in sign; hence, in numerous cases, one must choose other points than $2\pi\tau_k$ at which to test the sign of $F(t)$.

With regard to the calculation of $F(t)$, there are two methods which are used in the present work. The major portion of the computation was based on the Riemann-Siegel asymptotic formula; the error committed by taking only the first term has been numerically bounded by Titchmarsh, who gave usable estimates for almost all cases. It might be mentioned that Titchmarsh's bound is not quite correct and the present author does not indicate whether he used a corrected form of the bound. The author, however, gives three additional terms of the asymptotic expansion together with a fourth term which is written (with a slight change of notation) as $o(t^{-2})$. It is not entirely clear whether this extension has been used in the calculation, but there is evidence that it was. If so, we must express regret that the author did not replace the term $o(t^{-2})$ by a bound ct^{-2} with an explicit value for c; without such a value for c, one can not be entirely certain that the contribution of this term is not large enough to alter the sign of $F(t)$. In at least two cases where Titchmarsh's estimate was not strong enough to establish the sign of $F(t)$ recourse was made to an earlier method, based on the Euler-Maclaurin sum formula. This method has associated with it a smaller error term but suffers from the defect of requiring that a large number of terms (1500 and 2000) be taken in a sum approximating $F(t)$. We also note that the author makes no mention of the values t for which he has established that $\Re\zeta(\sigma+it)\neq 0$ for $\frac{1}{2}\leqq\sigma\leqq 2$ and, as a result, it is not clear for which values of t the quantity $N(t)$ was precisely determined.

The author gives statistics concerning the failures of Gram's law. In the last 15,000 intervals $(2\pi\tau_n, 2\pi\tau_{n+1})$, about 10% fail to contain a zero of $\zeta(\frac{1}{2}+it)$. However, all but about 1% of the intervals $(2\pi\tau_{n-1/4}, 2\pi\tau_{n+5/4})$ contain a zero. The total calculation took only a few hours of time on the computing machine known as SWAC.

L. Schoenfeld (East Pittsburgh, Pa.).

Hornfeck, Bernhard. Verallgemeinerte Primzahlsätze. Monatsh. Math. **60** (1956), 93–95.

For a positive integer d, the prime p is called d-isolated if it is the only prime in the interval $[p-d, p+d]$. W. Sierpiński [Colloq. Math. **1** (1948), 193–194; MR **10**, 431] proved that, for any given d, there exist d-isolated primes; and K. Prachar [Monatsh. Math. **58** (1954), 114–116; MR **16**, 114] sharpened this result by showing that almost all primes are d-isolated. In the present paper the author states the following more general theorem. Let $\mathfrak{A}$ be a set of positive and pairwise coprime integers, and suppose that the number of $a\in\mathfrak{A}$, $a\leqq x$, exceeds $cx/\log x$ for some $c>0$. Let $a\in\mathfrak{A}$ be called d-isolated if it is the only element of $\mathfrak{A}$ in the interval $[a-d, a+d]$. Then, for any given d, almost all elements of $\mathfrak{A}$ are d-isolated. It is shown that this result depends on a generalization of Theorem 88 in E. Landau's tract "Über einige neuere Fortschritte der additiven Zahlentheorie" [Cambridge, 1937]; for the proof of this generalization the reader is referred to J. Reine Angew. Math. **196** (1956), 156–169 [MR **18**, 564]. The author also states a theorem concerning the density of the set of numbers d for which the equation $d=a-a'$, $a\in\mathfrak{A}$, $a'\in\mathfrak{A}$ is soluble. This theorem generalizes an earlier result of Prachar [Monatsh. Math. **56** (1952), 304–306; MR **14**, 727.]

The factor x^2 in the equation (2) should be x.

L. Mirsky (Sheffield).

Hartman, S.; und Knapowski, S. Bemerkungen über die Bruchteile von $p\alpha$. Ann. Polon. Math. 3 (1957), 285–287.

The authors discuss the solubility of

$$(*)\qquad |\alpha-r/q|<B/q^b$$

in (positive) integers r, q, of which one or both may be required to be prime. Here α, B, b are given positive numbers (α irrational). In Satz 1 it is proved that, if c is a (Linnik) constant such that every arithmetical progression $l+jq$ ($j=1, 2, \cdots$; $0<l<q$; $(l, q)=1$) contains a prime $p<q^c$, and if α is such that (*) has an infinity of integer solutions with $(r, q)=1$ when $b=c+1$, $B=1$, then (*) has an infinity of solutions with q prime when $b=1+c^{-1}$, $B=2$. In Satz 2 the known fact that there exists a prime between x and $x+x^{5/8+\varepsilon}$ (ε an arbitrarily small positive number; $x>x_0(\varepsilon)$) is made to yield the consequence that (*) has an infinity of solutions with r, q both prime when $b=3/8-\varepsilon$, $B=1$. These theorems are elementary deductions from the basic facts about primes. The authors also quote an observation of Jarník that it

follows from a theorem of Vinogradov on the distribution of the fractional parts of $p\alpha$ [Trudy Mat. Inst. Steklov. **23** (1947), p. 177; MR **10**, 599; **15**, 941] that (*) has an infinity of solutions with q prime when $b=6/5-\varepsilon$, $B=1$.
A. E. Ingham (Cambridge, England).

Härtter, Erich. Ein Beitrag zur Theorie der Minimalbasen. J. Reine Angew. Math. **196** (1956), 170–204.

Let $M=\{x_0, x_1, x_2, \cdots, x_n, \cdots\}$ with $x_0=0$ and $x_n<x_{n+1}$ for $n=0, 1, 2, \cdots$ be a finite or infinite set of non-negative integers. A set $B=\{b_0, b_1, b_2, \cdots, b_k, \cdots\}$ of non-negative integers is called a basis of order h ($h\geqq 1$) for M, if for each number $x_n \in M$, a representation

$$x_n=\sum_{g=1}^{h} b^{(g)} \quad (b^{(g)} \in B)$$

exists. Clearly $0 \in B$. The case $M=Z$, the set of all non-negative integers, is of interest. An asymptotic basis of order h for a set M is defined in the natural way.

The "minimal base" concept stems from Rohrbach, Stöhr and Chatrovsky [see, e.g., Stöhr, same J. **194** (1955), 40–65, 111–140; MR **17**, 713]. These authors distinguish no less than nine different types of minimal bases. We give the definitions for some of these, followed by sample theorems of the author; condition (j) below is the requirement on $B(n)$ for a basis B of order h for M to be a minimal basis of the 'jth kind'. — (1) $B(n)\leqq\tilde{B}(n)$ ($n=1, 2, 3, \cdots$), where $\tilde{B}$ is an arbitrary basis of order h for M. (2) $B(n)\leqq\tilde{B}(n)$ for $n\geqq N$, N independent of $\tilde{B}$, where $\tilde{B}$ has the same meaning as in (1). (3) Same as (2) except that $N=N(\tilde{B})$ is allowed to depend on $\tilde{B}$. (4) $B(n_i)\leqq\tilde{B}(n_i)$ for an infinite sequence of natural numbers n_i. Here $\tilde{B}$ is arbitrary, h and M fixed. (8) $B(n)=O((M(n))^{1/h})$. If B is an asymptotic basis, we extend the above definitions and speak of asymptotic minimal bases of the 1st, 2nd, $\cdots$, 8th kinds.

The question whether for $M=Z$ there exist minimal bases of the second or third kinds of order $h\geqq 2$, is still unanswered. Sample theorems: (Satz 4) If minimal bases of the second or third kinds of order h exist, then the class of these minimal bases is countable. (Satz 16) For no set M do there exist asymptotic minimal bases of the fourth kind of order $h\geqq 1$. (Satz 41) If an infinite set M possesses minimal bases of type 8 (order h), then the class K_8^h of these minimal bases has the power of the continuum.
S. Chowla (Princeton, N.J.).

Erdös, P. Einige Bemerkungen zur Arbeit von A. Stöhr: "Gelöste und ungelöste Fragen über Basen der natürlichen Zahlenreihe". J. Reine Angew. Math. **197** (1957), 216–219.

The author gives another solution of a problem proposed by A. Stöhr [same J. **194** (1955), 40–65, 111–140; MR **17**, 713] and solved by E. Härtter [see preceding review]. He proves, namely, the theorem that the class of minimal bases of the 7th kind of order h ($h\geqq 2$) has the power of the continuum, the notation used being that of Stöhr. *S. Chowla* (Princeton, N.J.).

Stöhr, Alfred; und Wirsing, Eduard. Beispiele von wesentlichen Komponenten, die keine Basen sind. J. Reine Angew. Math. **196** (1956), 96–98.

As in the review of A. Stöhr [same J. **194** (1955), 40–65, 111–140; MR **17**, 713], let Z denote the set of all non-negative integers, A, B and W any sub-sets of Z. Let $A(x)$, $B(x)$, etc., denote the number of members of A, B, etc., which are >0 and $\leqq x$; $A+B$ is the set of all $a+b$ with $a \in A$, $b \in B$. Also write

$$\delta(A)=\inf \frac{A(n)}{n} \quad (n=1, 2, 3, \cdots).$$

W is called a "wesentliche Komponente" if from $0<\delta(A)<1$ it always follows that $\delta(A+W)>\delta(A)$. Erdös showed [Acta Arithmetica **1** (1936), 197–200] that every basis is a "wesentliche Komponente". The authors give a simpler example of a "wesentliche Komponente" which is not a basis, than the well-known but complicated one of Linnik [Mat. Sb. N.S. **10(52)** (1942), 67–78; MR **4**, 131].
S. Chowla (Princeton, N.J.).

Hornfeck, Bernhard; und Wirsing, Eduard. Über die schwache Basisordnung. Arch. Math. **7** (1957), 450–452.

Given a set S of non-negative integers; let h be the least integer with the property that all large integers are expressible as a sum of h members of S; let k be the least integer with the property that "almost all" integers (in the sense of Hardy and Littlewood) are sums of k members of S. It is easy to show that $k\leqq h\leqq 2k$. The authors show that h can assume any value between k and $2k$ (both these limits included), when the set S is suitably chosen. *S. Chowla* (Princeton, N.J.).

Kelly, John B. Restricted bases. Amer. J. Math. **79** (1957), 258–264.

A sequence of integers $1\leqq a_1<a_2<\cdots$ is called an additive basis of order k if every integer can be expressed as the sum of k or fewer of the a's. It is called an asymptotic basis of order k if every sufficiently large integer is the sum of k or fewer a's. It is called a restricted basis of order k if every sufficiently large integer is the sum of k or fewer distinct a's.

The author shows that a basis of order 2 is a restricted basis of order at most 4, and he states that perhaps an asymptotic basis of order 2 is a retricted basis of order at most 3, but he can only prove some special cases of this conjecture. A simple example of Bateman shows that a basis of order 3 does not have to be a restricted basis of any order ($a_1=1$, $a_2=3$, $a_3=6$, $\cdots$, $a_n=3(n-1)\cdots$).
P. Erdös (Toronto, Ont.).

★**Popken, J. Some theorems concerning transcendental numbers.** Colloque sur la Théorie des Nombres, Bruxelles, 1955, pp. 107–110. Georges Thone, Liège; Masson and Cie, Paris, 1956.

Some theorems are stated, the proofs of which appear in full in Bull. Soc. Math. Belg. **7** (1955), 124–130 [MR **18**, 566]. Main theorem: Let $F(z)$ denote a polynomial with algebraic coefficients $\not\equiv 0$ such that $F(0)=0$; moreover let r denote an arbitrary rational constant. Put for $n=1, 2, \cdots$

$$F_n=\begin{cases} \dfrac{1}{n+r}\left(\dfrac{d^n}{dz^n}\, e^{(n+r)F(z)}\right)_{z=0} & \text{if } n+r\neq 0, \\ 0 & \text{if } n+r=0. \end{cases}$$

Then $f(z)=\sum_{n=1}^{\infty} F_n z^n/n!$ generates a many-valued analytic function. For every algebraic z, distinct from the zeros of $F(z)$, the function $f(z)$ is regular, and $f(z)$ and all its derivatives take there transcendental values.
J. F. Koksma (Amsterdam).

Theory of Algebraic Numbers

Lehmer, Emma. On the location of Gauss sums. Math. Tables Aids Comput. **10** (1956), 194–202.

The generalized Gauss sum of order k,

$$S_k=\sum_{m=0}^{p-1}\exp(2\pi i m^k/p),$$

$p=kf+1$, p prime, satisfies $|S_k|\leqq(k-1)\sqrt{p}$ and its distribution in this interval is subject to conjecture. The various hypotheses are examined in detail for $k=3,4,5,7$ using numerical evidence obtained from the SWAC. The earlier computation of S_3 [von Neumann and Goldstine, same journal **7** (1953), 133–134; MR **14**, 1126] is extended to the 1000th prime of the form $6n+1$. For $k=4$ and 5 the results extend to $p<10{,}000$ and for $k=7$, $p<5000$. All the conjectures concerning non-uniform distribution are suspected to be false, and the author looks with favor upon the hypothesis that for odd k, $|S_k|$ is equally distributed in the intervals $(0, \sqrt{p})$, $(\sqrt{p}, \sqrt{(kp)})$, and $(\sqrt{(kp)}, (k-1)\sqrt{p})$.

J. L. Selfridge (Los Angeles, Calif.).

Goldberg, Karl. Unimodular matrices of order 2 that commute. J. Washington Acad. Sci. **46** (1956), 337–338.

It is proved that every abelian subgroup of the modular group is cyclic. The method of proof consists in transforming the unimodular 2×2 matrices of rational integers to canonical form by a similarity. The eigenvalues of such matrices are units in quadratic fields and the known structure of the group of units in such fields yields the result. [For other proofs and generalizations see the paper reviewed below.]

O. Taussky-Todd.

Taussky, Olga; and Todd, John. Commuting bilinear transformations and matrices. J. Washington Acad. Sci. **46** (1956), 373–375 (1957).

Another proof is given of the following theorem, proved in the paper reviewed above. A necessary and sufficient condition that two transformations of the type

$$w=(az+b)/(cz+d),$$

a, b, c, d rational integers, $ad-bc=1$, commute is that each is the iterate of the same transformation of this type. This theorem is extended to cover the case in which a, b, c, d are integers in a complex quadratic field F with determinant equal to a unit in F. This generalization is not immediate and in fact is subject to the following two provisos. The matrices of neither of the two transformations may be similar to $\zeta\begin{pmatrix}1&1\\0&1\end{pmatrix}$, where ζ is a root of unity in F, and the characteristic roots of neither matrix may be in the cyclotomic field generated by a primitive 8th or 12th root of unity. For example, the two matrices with Gaussian integer elements

$$\begin{pmatrix}0&1\\1&-i\end{pmatrix} \text{ and } \begin{pmatrix}2+i&2\\2&2-i\end{pmatrix}$$

commute and yet they are not powers of the same matrix.

D. H. Lehmer (Berkeley, Calif.).

Samko, G. P. Basis of a field determined by a root of prime odd degree of a rational integer. Rostov. Gos. Ped. Inst. Uč. Zap. no. 3 (1955), 29–37. (Russian)

Let $p>2$ be a prime, $t_1>0$ an integer not divisible by a pth power of an integer; $t_1=a_1a_2^2a_3^3\cdots a_{p-1}^{p-1}$ with $(a_i, a_k)=1$, $i\neq k$. Let $t_k=M_k^{-p}\cdot t_1^k$ $(k=2, \cdots, p-1)$, where M_k is an integer and t_k is not divisible by a pth power. A basis of integers of the field $R(t_1^{1/p})$ (R the rational number field) is obtained. It is proved: (1) If $t_1\equiv 0 \pmod{p}$ and $t_1\not\equiv 0 \pmod{p^2}$, or if $t_1\not\equiv 0 \pmod{p}$ and $t_1^p-t_1\not\equiv 0 \pmod{p^2}$, then $1, t_1^{1/p}, t_2^{1/p}, \cdots, t_{p-1}^{1/p}$ forms an integral basis. (2) If $t_1\not\equiv 0 \pmod{p}$ and $t_1^p-t_1\equiv 0 \pmod{p^2}$, then the numbers $1, t_1^{1/p}, t_2^{1/p}, \cdots, t_{p-2}^{1/p}, \sigma$ form an integral basis. Here

$$\sigma=\frac{1}{p}[u^{p-1}M+u^{p-2}Mt_1^{1/p} +u^{p-3}M_2Mt_2^{1/p}+\cdots+uM_{p-2}Mt_{p-2}^{1/p}+t_{p-1}^{1/p}],$$

where $M_{p-1}M\equiv 1 \pmod{p}$ and $u=t_1$ if $t_1^p-t_1\not\equiv 0 \pmod{p^3}$, $u=t_1+p$ if $t_1^p-t_1\equiv 0 \pmod{p^3}$. (The integers in the bracket can be, of course, reduced mod p.)

{Reviewers note: An exhaustive description of the construction of a basis of the field $R(a^{1/n})$ ($n>1$, a an integer) is given in W. E. H. Berwick, "Integral bases" [Cambridge, 1927, pp. 63–93]. The author considers the special case when n is a prime. This enables him to get an explicit solution and to avoid the construction of various partial bases modulo the prime factors of a and n.}

Š. Schwarz (Bratislava).

Eichler, Martin. Der Hilbertsche Klassenkörper eines imaginärquadratischen Zahlkörpers. Math. Z. **64** (1956), 229–242.

Let k be an imaginary quadratic field and $\mathfrak{o}$ the ring of integers of k. Let $\mathfrak{M}$ be the algebra of rational matrices of degree 2 and $\mathfrak{O}$ the ring of integral matrices in $\mathfrak{M}$. Let J be the set of isomorphisms φ of $\mathfrak{o}$ with subrings of $\mathfrak{O}$. Two isomorphisms φ, φ' in J are called equivalent if $\varphi'(x)=\varepsilon\varphi(x)\varepsilon^{-1}$ $(x\in\mathfrak{o})$, where ε is an element of determinant $+1$ in $\mathfrak{O}$. For any inversible α in $\mathfrak{M}$, let S_α be the corresponding homographic substitution on a complex variable τ; if $\varphi\in J$, the elements $S_{\varphi(x)}$ $(x\in\mathfrak{o})$ have the same fixed points, of which one, say $\tau(\varphi)$, is in the upper half-plane. If φ and φ' are equivalent, then $\tau(\varphi)$ and $\tau(\varphi')$ may be transformed into each other by an operation of the modular group Γ_1, whence $j(\tau(\varphi'))=j(\tau(\varphi))$, if j is the modular function. The common value of the $j(\tau(\varphi))$ for all φ in an equivalence class $\mathfrak{K}$ is called the class-invariant of $\mathfrak{K}$ and denoted by $j(\mathfrak{K})$. The main theorem of classical complex multiplication asserts that the field K obtained by adjunction to k of the numbers $j(\mathfrak{K})$ (for all classes $\mathfrak{K}$) is the absolute class field of k. The present paper gives a simplified proof of this result. Let Γ_n be the set of all S_α, for $\alpha\in\mathfrak{o}$, det $\alpha=n$ (where n is an integer >0); Γ_n decomposes into a finite number of cosets Γ_1S_i modulo the modular group Γ_1. The polynomial $\prod_i(X-j(S_i\tau))$ may be expressed as $F_n(X, j(\tau))$, where F_n is a polynomial in two letters with rational coefficients; moreover, the denominators of the coefficients of F_n involve only a finite number of primes (if p is a prime which is not involved in the denominators of the coefficients of the Fourier expansion of j, then p is not involved in the denominators of the coefficients of any one of the F_n). Except for perhaps a finite number of primes p, one has the congruences

$$F_p(X, X')\equiv(X-X'^p)(X^p-X') \pmod{p},$$

$$F_{p^2}(X, X')\equiv(X-X'^{p^2})(X^{p^2}-X')(X-X')^p \pmod{p}.$$

These are the only function-theoretic facts whose knowledge is required for the present proof. The algebraic character of the numbers $j(\mathfrak{K})$ follows from the fact that $F_n(j(\mathfrak{K}), j(\mathfrak{K}))=0$ whenever n is the norm of an element $\neq 0$ in $\mathfrak{o}$. The main tool in the proof consists in representing the ideal class group of k as a group of permutations of the equivalence classes $\mathfrak{K}$ in the following manner. Let $\mathfrak{K}$ be

any equivalence class, φ a representant of $\mathfrak{K}$ and $\mathfrak{a}$ an ideal in $\mathfrak{o}$; then $\varphi(\mathfrak{a})$ generates a left-ideal in $\mathfrak{O}$; the left-ideals in $\mathfrak{O}$ being principal, $\mathfrak{O}\varphi(\mathfrak{a})$ may be written as $\mathfrak{O}\alpha$ ($\alpha \in \mathfrak{O}$), and it is clear that α has the property that $\alpha\varphi(\mathfrak{o})\alpha^{-1}$ lies in $\mathfrak{O}$; we may further take α in such a way that det $\alpha>0$; the mapping $\varphi':x\rightarrow\alpha\varphi(x)\alpha^{-1}$ is then an element of J, and its class is easily seen to depend only on $\mathfrak{K}$ and on the ideal class of $\mathfrak{a}$; this makes the ideal class group of k operate on the set of equivalence classes $\mathfrak{K}$. It is easily seen that the ideal class group actually operates on the set of the class invariants $j(\mathfrak{K})$. The author proves that the permutations of the numbers $j(\mathfrak{K})$ which correspond in this manner to the ideal classes of k are induced by automorphisms of the field K. Using these automorphisms, it is proved that (with possibly a finite number of exceptions) a prime ideal $\mathfrak{p}$ of k whose ideal class is of order f decomposes into prime ideals of relative degree f in K, which proves that K is the absolute class field.

The author points out that his method does not yield the Artin reciprocity law for K, essentially because it does not allow us to distinguish between prime ideals of k which are conjugate to each other over the rationals.

The proof offered by the author may still be substantially simplified if one is willing to use the fact that a normal extension M of a field L of algebraic numbers is uniquely determined when one knows (with possibly a finite number of exceptions) all prime ideals of the first degree of L which split completely in M.

A certain number of corrections to the present paper have been published by the author in a subsequent note [see the paper listed below]. *C. Chevalley* (Paris).

Eichler, Martin. Berichtigung zu meiner Arbeit über den Hilbertschen Klassenkörper eines imaginärquadratischen Zahlkörpers. Math. Z. 65 (1956), 214.

See also: Lang, p. 174.

Geometry of Numbers

Barbour, J. M. A geometrical approximation to the roots of numbers. Amer. Math. Monthly 64 (1957), 1–9.

The author considers various methods of approximation to the numbers $2^{r/12}(0<r<12)$ which have been proposed since 1581 [V. Galilei, Dialogo della musica antica e moderna, Florence, 1581], this being of interest as a practical method of achieving equal temperament of the musical scale. An ingenious geometric method put forward by Strähle [Kongl. Swenska Wetenskaps Acad. Handl. 4 (1743), 281–286] has not received its due credit since, when its exact implications were worked out by Faggot [ibid. 4 (1743), 286–291], he made a serious numerical error which implied that the maximum error of the approximation was about 1.7% instead of the correct amount of 0.15%. The author corrects this error and generalizes Strähle's method to give approximation to fractional powers N^m of any positive number N. The method consists essentially of replacing N^m by a bilinear transformation in m with coefficients adjusted to give the correct values at $m=0$, $\frac{1}{2}$, and 1, namely

$$N^m=\frac{Nm+N^{\frac{1}{2}}(1-m)}{m+N^{\frac{1}{2}}(1-m)}.$$

R. A. Rankin (Glasgow).

Lekkerkerker, Cornelis Gerrit. Una questione di approssimazione diofantea e una proprietá caratteristica dei numeri quadratici. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 21 (1956), 179–185.

Following Cugiani [Boll. Un. Mat. Ital. (3) 10 (1955), 489–497; MR 17, 829] the author considers the set $I(\alpha)$ of values of $r^2-s^2\alpha$, where $\alpha>0$ is a fixed irrational and r, s run through all integers. He puts $\alpha=\theta^2$ and considers also the much more natural set $J(\theta)$ of values of $s(s-r\theta)$. In an obvious sense it follows at once that $I'(\alpha)=2\theta J'(\theta)$, where $I'(\alpha)$, $J'(\theta)$ are the derived sets. The author now shows easily that $J'(\theta)$ is discrete if and only if θ is a quadratic irrationality. Using a result of Marshall Hall on the expression of a number as the sum of two numbers with bounded partial quotients [Ann. of Math. (2) 48 (1947), 966–993; MR 9, 226], he constructs a number θ whose partial quotients are at most $k-1$, where k is any integer greater than 6, and such that $J(\theta)$ is dense in the interval $(-1/k, 1/k)$. *J. W. S. Cassels.*

Perron, Oskar. Ein neuartiges diophantisches Problem. Math. Z. 67 (1957), 176–180.

Let a, b denote two different real numbers and let

$$f(x)=|a-x|\ |b-x|,$$

where x is a rational integral variable. In considering the minimum of $f(x)$ as x varies it is sufficient to take $0<a<1$, $a<b$. Then for the particular range $c=b-a\leqq\sqrt{5}$, the expected inequality $f(x)\leqq1$ is shown to be soluble, with strict inequality except in the case $a=\frac{1}{2}(3-\sqrt{5})$, $c=\sqrt{5}$. The same inequality is established later for the extended range $\sqrt{5}<c\leqq\sqrt{8}$. [For another proof, see C. S. Davis, Quart. J. Math., Oxford Ser. (2) 1 (1950), 241–242; MR 12, 393.] If now a, b denote complex numbers and $c=|b-a|$, the inequality $f(x)\leqq1$, where x is a gaussian integer, is stated to be soluble for $c<\sqrt{3}$. *J. H. H. Chalk.*

★ **Hlawka, Edmund. Das inhomogene Problem in der Geometrie der Zahlen.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 20–27. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

Let T be a set of points in n dimensional space and let $\mathcal{G}$ be a lattice in the space. If for every point p of space there is a lattice-vector g such that $p+g$ lies in T, we say that $\mathcal{G}$ is a covering lattice for T. Many problems, both of a general and a special nature, can be related to this concept, and the author's extensive survey includes much work that was previously scattered and unconnected. His collection of references will be valuable to all workers in the field. A few more recent references are: B. J. Birch, Proc. Cambridge Philos. Soc. 53 (1957), 269–272; J. W. S. Cassels, Introduction to Diophantine approximation, Cambridge, 1957; C. A. Rogers, Mathematika 4 (1957), 1–6; G. L. Watson, Rend. Circ. Mat. Palermo (2) 5 (1956), 93–100 [MR 17, 1235].

H. Davenport (London).

★ **Minkowski, Hermann. Diophantische Approximationen. Eine Einführung in die Zahlentheorie.** Chelsea Publishing Co., New York, 1957. viii+235 pp. $4.50.

This is a reprint of Minkowski's classical book [Teubner, Leipzig, 1907] with some minor corrections. The book was written to give an elementary exposition of his work on the Geometry of Numbers and of its applications to the

theories of Diophantine Approximation and of Algebraic Numbers. The work is now mainly interesting for historical reasons and for some of its details. It seems irritating nowadays to find general n-dimensional theorems proved first in 2dimensions and then again in later chapters in 3dimensions and in 4dimensions. Readers interested in finding the way the subject has developed are referred to the books: J. F. Koksma, Diophantische Approximationen [Springer, Berlin, 1936]; O.-H. Keller, Geometrie der Zahlen, Enzykl. Math. Wiss. I 2, 27 [Teubner, Leipzig, 1954; MR **16**, 451]; J. W. S. Cassels, Introduction to Diophantine approximation, [Cambridge, 1957]; and to the review articles: H. Davenport, Proc. Internat. Congress Math., Cambridge, Mass., 1950, v. 1, pp. 166–174 [Amer. Math. Soc., Providence, R. I., 1952; MR **13**, 919]; E. Hlawka, Jber. Deutsch. Math. Verein. **57** (1954), Abt. 1, 37–55 [MR **16**, 117]; and the paper reviewed above.

C. A. Rogers (Birmingham).

Inkeri, K.; and Ennola, V. The Minkowski constants for certain binary quadratic forms. Ann. Univ. Turku. Ser. A. **25** (1957), 19 pp.

If $f(x, y)=ax^2+bxy+cy^2$ is a real indefinite quadratic form with discriminant $d=b^2-4ac>0$ and, for any real numbers x_0, y_0, the lower bound of $|f(x+x_0, y+y_0)|$ taken over all integer sets x, y is denoted by $M(f; x_0, y_0)$, it is known by a theorem of Minkowski that $M(f; x_0, y_0)\leqq\frac{1}{4}\sqrt{d}$. The Minkowski constant $M(f)$ is defined to be the upper bound of $M(f; x_0, y_0)$ taken over all real sets x_0, y_0. By two slightly differing methods, the authors determine the values of $M(f_m)$ for the principal forms:

$$f_m(x, y)= \begin{cases} x^2-my^2 & (m=7, 11, 19), \\ x^2+xy-\frac{1}{4}(m-1)y^2 & (m=41). \end{cases}$$

Proofs for the cases $m=7$, 11 had been found previously by Inkeri (unpublished) and others, and for the cases $m=19$ (unpublished), 41 by Barnes and Swinnerton-Dyer [for details and full references see Acta Math. **87** (1952), 259–323; MR **14**, 730]. The case $m=19$ was considered later by Barnes [ibid. **92** (1954), 235–264; MR **16**, 802].

J. H. H. Chalk (Hamilton, Ont.).

Birch, B. J. A grid with no split parallelepiped. Proc. Cambridge Philos. Soc. **53** (1957), 536.

It has been shown by B. N. Delone [Izv. Akad. Nauk SSSR. Ser. Mat. **11** (1947), 505–538; MR **9**, 334] that every 2-dimensional grid (inhomogeneous lattice) contains a parallelogram with one vertex in each quadrant and of area equal to the determinant of the grid. This has been exploited by Delone himself (loc. cit.), by Barnes and Swinnerton-Dyer [Acta Math. **93** (1954), 199–234; MR **16**, 802] and Barnes [ibid. **92** (1954), 235–264; J. London Math. Soc. **31** (1956), 73–76; MR **16**, 802; **17**, 715]. The author gives an elegant counterexample to show that no generalization to 3 dimensions is possible.

J. W. S. Cassels (Cambridge, England).

Davenport, H. Note on a theorem of Cassels. Proc. Cambridge Philos. Soc. **53** (1957), 539–540.

It is shown that if the quadratic form $\sum\sum_{1\leq i,j\leq n} f_{ij}x_ix_j$ with integer coefficients represents 0 at all, then there is a representation with

$$0<\sum x_i^2\leq\gamma_{n-1}^{\,n-1}\Big(2\sum_{i,j} f_{ij}^2\Big)^{(n-1)/2},$$

where γ_{n-1} is Hermite's constant, in the usual notation, related to the minimum of positive definite quadratic forms in $n-1$ variables. Apart from the constant $\gamma_{n-1}^{\,n-1}$ this is equivalent to a result of the reviewer [same Proc. **51** (1955), 262–264; **52** (1956), 602; MR **16**, 1002; **18**, 380], but the author's proof is more elementary, since it does not require the theory of "successive minima".

J. W. S. Cassels (Cambridge, England).

Birch, B. J. Another transference theorem of the geometry of numbers. Proc. Cambridge Philos. Soc. **53** (1957), 269–272.

Let λ and Λ be respectively the homogeneous and the inhomogeneous minimum of a convex body of volume V with respect to a lattice of unit determinant. The author gives a short and elegant proof that $\Lambda^{n-1}\lambda V\geqq 2/n$ and a rather longer proof of a slightly stronger result. He shows by an example that these results cannot be much improved. The proofs are entirely from first principles. Similar but weaker results have been given by Mahler using "successive minima" [Časopis Pěst. Mat. Fys. **68** (1939), 93–102; MR **1**, 202].

J. W. S. Cassels.

See also: Hartman and Knapowski, p. 121; Rogers, p. 127; Woods, p. 164.

ANALYSIS

Functions of Real Variables

Zubov, V. I. On the question of existence and approximate representation of implicit functions. Vestnik Leningrad. Univ. **11** (1956), no. 19, 48–54. (Russian)

The author obtains a version of the implicit function theorem (for functions of several variables satisfying a system of simultaneous equations). His procedure is to obtain a system of partial differential equations of the first order and degree and then to construct solutions of these differential equations by an iterative method. His results are too complicated for reproduction here. He considers various examples showing that in some, but not all, cases the range of validity of his results is greater than that of the usual implicit function theorem [cf. E. Goursat, Cours d'analyse mathématique, vol. 1, 3rd ed., Gauthier-Villars, Paris, 1917, §§ 34, 35].

H. P. Mulholland (Birmingham).

Bruhat, François; et Cartan, Henri. Sur la structure des sous-ensembles analytiques réels. C. R. Acad. Sci. Paris **244** (1957), 988–990.

In a real Euclidean space let E be the set of zeros common to finitely many real-analytic functions, let p be its topological dimension and let $V_p(E)$ be the subset of E consisting of those points at each of which the dimension p is attained. The authors apply local complexification, and although the results are what one would expect the reasoning is not necessarily obvious. The main statement is as follows. If at a point a of E the local germ of real analytic functions defining E is irreducible and has "algebraic" dimension p, then in a certain neighborhood U the set $U\cap V_p(E)$ decomposes into finitely many connected sets A_i, and a is strongly adherent to each. That is, there is an arc γ_i emanating from a such that $\gamma_i-\{a\}\subset A_i$.

S. Bochner (Princeton, N.J.).

Aczél, János. Zur Theorie der Mittelwerte. Acta Univ. Debrecen. **1** (1954), 117–135; additamentum ad **1** (1955), 18. (Hungarian. German summary)

Dies ist ein zusammenfassender Artikel über die Resultate in der Mittelwerttheorie seit 1930, der als neues Resultat die allgemeinste streng monotone und zweimal stetig differenzierbare Lösung

$$M(x, y)=f^{-1}[pf(x)+(1-p)f(y)]$$

($0<p<1$, f streng monoton und zweimal stetig derivierbar mit der Inversen f^{-1}: quasilineares Mittel) der Autodistributivitätsgleichung

$$M[M(x, y), t]=M[M(x, t), M(y, t)]$$

enthält. Eine russische Fassung der Arbeit ist in Colloq. Math. **4** (1956), 33–55 [MR **18**, 876] erschienen.

Aus der Zusammenfassung des Autors.

Korányi, A. Note on the theory of monotone operator functions. Acta Sci. Math. Szeged **16** (1955), 241–245.

A function $f(x)$ is monotone of arbitrarily high order in $(-1, 1)$ if and only if it is analytic in $(-1, 1)$, can be analytically continued onto the entire upper half-plane, and has there a non-negative imaginary part. Bendat and Sherman [Trans. Amer. Math. Soc. **79** (1955), 58–71; MR **18**, 588] proved this theorem of the reviewer by use of the Hamburger moment problem which, with the normalization $f(0)=0$, leads to an integral representation

$$f(z)=\int_{-1}^{1} z(1-tz)^{-1}d\alpha(t)$$

with a non-decreasing, bounded function $\alpha(t)$. The author gives now a simple proof of the fact that the class of functions which allow such an integral representation is identical with the class of functions characterized above as functions monotonic of arbitrarily high order.

C. Loewner (Palo Alto, Calif.).

See also: Sargent, p. 126; Vasilach, p. 154; Massaro, p. 156.

Measure, Integration

Sikorski, R. On the Vitali theorem. Prace Mat. **2** (1956), 146–151. (Polish. Russian and English summaries)

The well-known theorem of Vitali on the covering of sets up to a set of Lebesgue measure 0 is generalized for other measures μ which are countably additive and satisfy the following conditions: (1) μ is defined in a countably additive ring of sets $\mathfrak{M}$ containing all Borel sets; (2) a set $A \in \mathfrak{M}$ if and only if there exist Borel sets A_1 and A_2 such that $A_1 \subset A$, $A-A_1 \subset A_2$, $\mu(A_2)=0$; (3) $\mu(A)<\infty$ for all bounded $A \in \mathfrak{M}$. This theorem is proved, for the one-dimensional case, in a very concise fashion, forming a modification of the well-known proof by Banach of the original Vitali theorem.

Some consequences of this theorem, for example, the theorem on the existence almost everywhere of a derivative of a function of bounded variation, follow in the same manner from Vitali's theorem as in the case of Lebesgue measure.

S. M. Ulam (Los Alamos, N.M.).

Sargent, W. L. C. On some cases of distinction between integrals and series. Proc. London Math. Soc. (3) **7** (1957), 249–264.

Two theorems are proved: (1) If the first of the following two integrals

$$\int_1^\infty x(t)k(t)dt, \quad \int_1^\infty x(t)dt$$

is bounded (C, μ) whenever the second is summable $|C, \lambda|$, $0\leq\mu<\lambda$, then for some real number $c\geq 1$ (i) $k(t)$ is measurable and essentially bounded in $(1, c)$, (ii) $k(t)=0$ p.p. in (c, ∞). (2) If the first of these integrals is summable $|C, \mu|$ whenever the second is summable (C, λ), $0\leq\mu<\lambda+1$, then for some real $c\geq 1$ (i) $k(t)$ is measurable and essentially bounded in $(1, c)$, (ii) $k(t)=0$ p.p. in (c, ∞).

These are analogues of results on series proved by Bosanquet [J. London Math. Soc. **20** (1945), 39–48, Th. 1, 2, 3; MR **7**, 432].

The proofs in the paper under review are obtained by a delicate analysis involving several lemmas which are interesting in themselves. Let $V_\lambda{}^c$ be the normed vector space which consists of functions $x(t)$ such that

$$x(t)=0,\ t<c, \text{ and } g(t)=tx(t) \in L(c, \infty),$$

the norm being defined by

$$||x||=\int_1^\infty t^{-(\lambda+1)}|G_\lambda(t)|dt,$$

where $G_\lambda(t)=(\Gamma(\lambda))^{-1}\int_1^t (t-u)^{\lambda-1}g(u)du$ $(\lambda>0)$ and $G_\lambda(t)=g(t)$ $(\lambda=0)$. B_λ, S_λ, V_λ are spaces of functions $x(t)$ such that $\int_1^\infty x(t)dt$ is respectively bounded (C, λ), summable (C, λ), and summable $|C, \lambda|$.

Lemma 1. If the first of the integrals,

$$\int_c^d x(t)\theta(t)dt, \quad \frac{t}{\Gamma(\lambda)}\int_t^\infty (u-t)^{\lambda-1}h(u)du,$$

is linear in $V_\lambda{}^c$, $\lambda>0$, $c<d<\infty$, then $\theta(t)$ is equal to the second p.p. in (c, d), where $t^{\lambda+1}h(t) \in M(c, \infty)$ (space of functions measurable and essentially bounded on (c, ∞)).

Lemma 2. If $\lambda<\mu$ and $x \in B_\lambda$, then there are real H and K independent of v and w such that when $1\leq w<v$

$$\left|\int_1^w (w-t)^\lambda(v-t)^{\mu-1}g(t)dt\right|,$$

$$\left|\int_1^w (w-t)^{\mu-1}\{(v-t)^\lambda-(v-w)^\lambda\}g(t)dt\right|$$

are respectively less than or equal to

$$H\, w^{\lambda+1}\{v^{\mu-1}+(v-w)^{\mu-1}\},$$
$$Kw^{\mu-1}\{v^{\lambda-1}+(v-w)^{\lambda-1}\}.$$

Lemma 3. If $\lambda=n+p$, where n is zero or a positive integer and $0<p\leq 1$, and if the function $\theta(t)$ is given by

$$\frac{1}{\Gamma(\lambda)}\int_t^\infty (u-t)^{\lambda-1}f(u)du, \text{ then } \frac{(-1)^n}{\Gamma(p)}\int_t^\infty (u-t)^{p-1}f(u)du$$

represents $\theta^{(n)}(t)$, $t\geq c$, where only the right hand derivative is considered for $t=c$.

Lemma 4. If $xk \in B_\mu$ whenever $x \in V_\lambda$ $(\lambda>0)$, then for some real number $c\geq 1$ (i) $k \in M(1, c)$; (ii) $k(t)$ is given by

$$\frac{t}{\Gamma(\lambda)}\int_t^\infty (\mu-t)^{\lambda-1}h(u)du,$$

where $t^{\lambda+1}h(t) \in M(c, \infty)$; and (iii)

$$\frac{1}{d^\mu}\int_c^d (d-t)^\mu x(t)k(t)dt$$

is a linear functional in $V_\lambda{}^c$.

Lemma 5. If $\lambda<\mu$, $xk \in V_\mu$ whenever $x \in S_\lambda$, then (i) $k \in M(1, \infty)$, (ii) $t^{-(\mu+1)}k(t)G_\mu(t) \in L(1, \infty)$ whenever $x \in S_\lambda$.

R. L. Jeffery (Kingston, Ont.).

Rogers, C. A. A single integral inequality. J. London Math. Soc. 32 (1957), 102–108.

When ρ is a bounded non-negative Lebesgue-integrable function on Euclidean n-space, the spherical symmetrization ρ^* of ρ is the function defined for each point X in n-space (except $X=0$, at which point ρ^* is defined to be $\sup\{\rho(X):X \text{ in } n\text{-space}\}$) as the supremum of numbers t such that the measure of the set of points Y with $\rho(Y)>t$ is not less than the measure of the set of points Y with $|Y|<|X|$. Generalizing results of an earlier paper of his [same J. 31 (1956), 235–38; MR 18, 757], the author proves the following theorem: if $\rho_1, \cdots, \rho_k$ are bounded non-negative Lebesgue-integrable functions and $\rho_1^*, \cdots, \rho_k^*$ are their respective spherical symmetrizations, then for any constants c_{ij} ($i=1, \cdots, k$; $j=1, \cdots, m$),

$$\int\cdots\int \prod_{i=1}^{k} \rho_i\Big(\sum_{j=1}^{m} c_{ij}X_j\Big)dX_1\cdots dX_m \leq \int\cdots\int \prod_{i=1}^{k}\rho_i^*\Big(\sum_{j=1}^{m} c_{ij}X_j\Big)dX_1\cdots dX_m.$$

An application of interest in the geometry of numbers is noted. *T. A. Botts* (Charlottesville, Va.).

Pistoia, Angelo. Estensione agli integrali del metodo di sommazione di Nörlund e teoremi di composizione. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 21 (90) (1956), 27–77.

In questo lavoro si introduce un procedimento d'integrazione (ad una o due dimensioni) "generalizzata", analogo al procedimento di Nörlund di sommazione delle serie, e si dimostrano svariati teoremi "di permanenza" e "di composizione" utili nella teoria della trasformazione di Laplace, e nei quali rientrano come casi particolari analoghi teoremi ottenuti in precedenza da A. Ghizzetti [Univ. e Politec. Torino. Rend Sem. Mat. 9 (1950), 251–261; MR 12, 498] e dall'A. stesso [Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 18(87) (1954), 627–652; MR 17, 480] mercè l'impiego d'un metodo d'integrazione generalizzata "alla Cesàro", più particolare di quello qui definito. Riferiamo, per dare un'idea del carattere di questi risultati, i due teoremi più semplici, riferentisi agli integrali ad una dimensione.

Data una funzione $\varphi(x)$ integrabile in ogni intervallo limitato della semiretta $x\geqq 0$ e diversa da zero in un intorno di $+\infty$, diremo "integrabile φ in $0+\infty$" (in simboli: $\epsilon\mathfrak{J}_\varphi$) una funzione $F(x)$ che sia integrabile in ogni intervallo limitato della semiretta $x\geqq 0$, e tale che esista finito il limite $I_\varphi(F)=\lim_{x\to+\infty}[F*\varphi(x)/\varphi(x)]$; diremo che due funzioni $\varphi(x)$ e $\psi(x)$ sono in "relazione Γ_1" (in simboli: $\varphi\Gamma_1\psi$) quando: (a) esse sono integrabili in ogni intervallo limitato della semiretta $x\geqq 0$; (b) per un certo $\sigma>0$ ed un certo $L>0$, e per ogni $x\geqq 2\sigma$ si ha $\varphi*\psi(x)\neq 0$ e

$$\int_\sigma^{x-\sigma} |\varphi(\tau)\psi(x-\tau)|d\tau/|\varphi*\psi(x)|\leqq L;$$

(c) per ogni $\tau\geqq 0$ si ha

$$\lim_{x\to+\infty}[\varphi(x-\tau)/\varphi*\psi(x)]=\lim_{x\to+\infty}[\psi(x-\tau)/\varphi*\psi(x)]=0;$$

(d) per ogni $T\geqq 0$ vi sono due numeri $M_T>0$ ed $N_T\geqq T$ tali da aversi

$$|\varphi(x-\tau)|\leqq M_T|\varphi*\psi(x)| \text{ e } |\psi(x-\tau)|\leqq M_T|\varphi*\psi(x)|$$

per $x\geqq N_T$ e $0\leqq\tau\leqq T$. Ciò premesso, si hanno i seguenti teoremi: (i) date le funzioni F, λ, φ, se è $F\,\epsilon\,\mathfrak{J}_\varphi$ ed esiste una funzione ψ per cui sia $\lambda=\varphi*\psi$ e $\varphi\Gamma_1\psi$, allora è di conseguenza $F\,\epsilon\,\mathfrak{J}_\lambda$ ed $I_\lambda(F)=I_\varphi(F)$ (teorema "di permanenza"); (ii) date le funzioni F, G, φ, ψ e $\lambda=\varphi*\psi$, se è $F\,\epsilon\,\mathfrak{J}_\varphi$, $G\,\epsilon\,\mathfrak{J}_\psi$, $\varphi\Gamma_1\psi$, allora si ha di conseguenza $F*G\,\epsilon\,\mathfrak{J}_\lambda$, $I_\lambda(F*G)=I_\varphi(F)\cdot I_\psi(G)$ (teorema "di composizione"). I citati teoremi di A. Ghizzetti si possono riottenere per $\varphi=x^\alpha$, $\psi=x^\beta$, $\alpha>-1$, $\beta>-1$. *F. Bertolini.*

Morse, Marston; and Transue, William. C-bimeasures Λ and their superior integrals Λ^*. Rend. Circ. Mat. Palermo (2) 4 (1955), 270–300 (1956).

A direct sequel to recent papers by the same authors [Morse and Transue, J. Analyse Math. 4 (1954/55), 149–186; Morse, Ann. Mat. Pure Appl. (4) 39 (1955), 345–356; MR 17, 469, 720]. Henceforth we adopt the notation used in the reviews just cited. In the main the present paper sets forth details and proofs of matters discussed in the second paper referred to.

In § 2 one finds proved for C-bimeasures on $E'\times E''$ several well-known properties of bilinear forms on products of topological vector spaces. The set of bounded C-bimeasures is formed into a Banach space in the usual fashion.

§§ 3–7 deal with the "variation" Λ^* of a bimeasure Λ on $E'\times E''$, which is defined by analogy with the variation $|\alpha|^*$ of a (Radon) measure α. Thus, if $\mathscr{I}_+'$ (resp. $\mathscr{I}_+''$) is the set of positive lower semicontinuous functions on E' (resp. E''), $\Lambda^*(p, q)$ is defined for $(p, q)\,\epsilon\,\mathscr{I}_+'\times\mathscr{I}_+''$ as $\sup\{|\Lambda(u, v)|:|u|\leqq p, |v|\leqq q\}$, where u (resp. v) belongs to $\mathscr{K}_C'$ (resp. $\mathscr{K}_C''$). For arbitrary positive functions h (resp. k) on E' (resp. E''), $\Lambda^*(h, k)$ is defined to be $\inf\{\Lambda^*(p, q):p\geqq h, q\geqq k\}$, where $(p, q)\,\epsilon\,\mathscr{I}_+'\times\mathscr{I}_+''$. It appears (Theorem 3.1) that Λ^* is subadditive in each argument; unlike $|\alpha|^*$, additivity in each argument usually fails. For $(p, q)\,\epsilon\,\mathscr{I}_+'\times\mathscr{I}_+''$, $\Lambda^*(p, q)$ is the limit, or supremum, of $\Lambda^*(u, v)$ when u (resp. v) ranges over any increasing directed set H_p' (resp. H_q'') in $\mathscr{K}_+'$ (resp. $\mathscr{K}_+''$) whose limit is p (resp. q) (Theorem 4.1).

Two limit theorems for Λ^* are given (Theorems 6.1 and 7.1). The first reads: If

$$(p, q)\,\epsilon\,\mathscr{I}_+'\times\mathscr{I}_+'' \text{ and } \Lambda^*(p, q)<+\infty,$$

then

$$\inf_{f\epsilon H_p'}\Lambda^*(p-f, q)=\lim_{f\epsilon H_p'}\Lambda^*(p-f, q)=0,$$

and analogously for the second argument; and the second: If $\Lambda^*(h, k)<+\infty$, then

$$\lim_{p'\epsilon+\varphi_h'}\Lambda^*(p-p', k)=\lim_{q'\epsilon+\varphi_k''}\Lambda^*(h, q-q')=0$$

holds if p' (resp. q') is in some section of $_+\varphi_h'$ (resp. $_+\varphi_k''$), where $_+\varphi_h'$ (resp. $_+\varphi_k''$) is the decreasing directed set of $p\,\epsilon\,\mathscr{I}_+'$ (resp. $q\,\epsilon\,\mathscr{I}_+''$) majorising h (resp. k).

The remaining §§ 8–11 deal with the case in which E' and E'' are real intervals (this is the classical case due to Fréchet), and with counterexamples illustrating the general theory. *R. E. Edwards* (London).

Morse, Marston; and Transue, William. C-bimeasures Λ and their integral extensions. Ann. of Math. (2) 64 (1956), 480–504.

This paper continues the investigations begun in several recent publications of the same authors [see especially J. Analyse Math. 4 (1954/55), 149–186; MR 17, 469; and also the paper reviewed above].

With the notation of earlier reviews, the central problem appears to be that of extending a bimeasure Λ from $\mathscr{K}_C'\times\mathscr{K}_C''$ to a wider class of functions. It is stated that this may be done in four stages, the order of which may be

varied. Consequently there arise questions of "commutativity" of the component stages, and also the matter of analogues of the Fubini theorem for product measures. These questions are examined in considerable detail, use being made of the Radon measures associated with Λ by fixing one of the two variable functions, and of the superior integral Λ^* introduced in earlier papers.

A specimen result is as follows. If $(u, v) \in \mathscr{K}_C' \times \mathscr{K}_C''$, $\Lambda(u, \cdot)$ is the measure on E'' defined by $v \to \Lambda(u, v)$, and $\Lambda(\cdot, v)$ is defined analogously. If $(x, y) \in C^{E'} \times C^{E''}$, $\Lambda(x, \cdot)$ is said to exist if (a) $\Lambda(\cdot, v)(x)$ exists for each $v \in \mathscr{K}_C''$ and (b) $v \to \Lambda(\cdot, v)(x)$ is a measure on E''; this measure is then by definition $\Lambda(x, \cdot)$. $\Lambda(\cdot, y)$ is defined similarly. A pair (x, y) is said to be Λ-integrable if the measures $\Lambda(x, \cdot)$ and $\Lambda(\cdot, y)$ exist, and if the integrals $\Lambda(x, \cdot)(y)$ and $\Lambda(\cdot, y)(x)$ exist and are equal. One has then the result (Theorem 7.1): If $\Lambda(x, \cdot)(y)$ and $\Lambda(\cdot, y)(x)$ exist, and if $\Lambda^*(|x|, |y|) < +\infty$, then (x, y) is Λ-integrable.

Counterexamples are given to show that $\Lambda(x, \cdot)(y)$ and $\Lambda(\cdot, y)(x)$ may exist and be unequal; and that the condition $\Lambda^*(|x|, |y|) < +\infty$ is not necessary in order that (x, y) be Λ-integrable. These points are further clarified in the special case in which $x = p \in \mathscr{I}_+'$ and $y = q \in \mathscr{I}_+''$.

Finally, the Λ-integrability of Borel pairs (x, y) is discussed in some detail. *R. E. Edwards* (London).

LeCam, Lucien. Convergence in distribution of stochastic processes. Univ. Calif. Publ. Statist. 2 (1957), 207–236.

Dans la première partie de ce travail l'auteur considère la situation suivante: X est un ensemble, H une algèbre de fonctions bornées (réelles) sur X, complète pour la norme usuelle de la convergence uniforme, et séparant les points de X. On considère sur X la structure uniforme la moins fine rendant uniformément continues les fonctions de H; en complétant X pour cette structure, on immerge X dans un espace compact T, où X est partout dense. Cela étant, l'auteur classe les mesures sur T par leur comportement relatif à H, de la façon suivante: comme en général T n'est pas métrisable, il y a deux sortes de "mesure extérieure" associées à une mesure $\mu \geq 0$, que l'auteur note μ^* et $\bar{\mu}$ [la seconde étant la mesure notée μ^{**} dans Bourbaki, Eléments de mathématique, XIII, Actualités Sci. Ind., no. 1175, Hermann, Paris, 1952, chap. IV, § 1, no. 5; MR 14, 960]; corrélativement on a deux mesures intérieures μ_* et $\mu_- \leq \mu_*$. L'auteur dit alors qu'une mesure $\mu \geq 0$ sur T est serrée ("tight") si $T - X$ est de mesure nulle, τ-régulière ("τ-smooth") si $\mu_*(T - X) = 0$ et σ-régulière ("σ-smooth") si $\mu_-(T - X) = 0$; ces deux dernières conditions équivalent, comme le montre l'auteur, aux conditions (MA) et (MA') (loc. cit.) lorsque μ est considérée comme "mesure abstraite" sur H; pour une mesure μ de signe quelconque, on dit que μ a l'une des trois propriétés précédentes lorsque $|\mu|$ a cette propriété. L'auteur est principalement intéressé en des conditions assurant qu'une limite d'une suite ou d'un filtre de mesures pour la topologie de la convergence simple dans H ait l'une des propriétés précédentes, ou pour qu'un ensemble de mesures soit compact pour cette topologie; il indique un certain nombre de tels critères, en partie déjà connus, et que nous ne pouvons résumer ici. Il applique ces idées dans la seconde partie à des généralisations du théorème de P. Lévy sur les "fonctions caractéristiques" des probabilistes (i.e. les transformées de Fourier de mesures bornées). Il commence par définir cette notion de façon très générale: si X est un espace localement convexe, X^* son dual, tout caractère sur le groupe topologique X peut s'écrire χ_t: $x \to \exp(i\langle t, x\rangle)$, où $t \in X^*$; la transformée de Fourier d'une mesure μ sur X est alors la fonction $t \to \hat{\mu}(t) = \mu(\chi_t)$ sur X^*. Ceci est appliqué uniquement au cas où $\mu = P$ est définie sur $H = C(X)$, espace des fonctions continues bornées sur X, et a l'une des trois propriétés étudiées plus haut. Les questions étudiées concernent alors des critères déduisant des propriétés de convergence d'une suite (P_n) de telles mesures, de propriétés de convergence de leurs transformées de Fourier $\hat{P}_n$. *J. Dieudonné* (Evanston, Ill.).

Functions of Complex Variables

Rizza, Giovanni Battista. Teoremi e formule integrali nelle algebre di Clifford. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 20 (1956), 174–178.

This report is an abstract of the author's detailed exposition which appeared in Rend. Mat. e Appl. (5) 15 (1956), 53–79 [MR 18, 725]. For related results see Takasu, Yokohama Math. J. 1 (1953), 131–224; 2 (1954), 1–68, 107–126 [MR 16, 350; 17, 28, 1193], and also Nisigaki and Takasu, ibid. 3 (1955), 53–126 [MR 18, 725].
T. Takasu (Yokohama).

Fuchs, W. H. J. A theorem on power series whose coefficients have given signs. Proc. Amer. Math. Soc. 8 (1957), 443–449.

The well-known theorem of Hurwitz and Pólya [Acta Math. 40 (1915), 179–183] says that if $\sum a_k z^k$ is a power series with finite radius of convergence then it is possible to find a sequence $\{e_k\}$ $(e_k = \pm 1)$ such that the series $\sum e_k a_k z^k$ has the circle of convergence as natural boundary. In the present paper the author proves the following sort of converse to the above result. If $\{e_k\}$ is a sequence with $e_k = \pm 1$ then there is always a power series $\sum a_k z^k$, $a_k > 0$, of finite radius of convergence such that the series $\sum e_k a_k z^k$ can be analytically continued across a semi-circle on its circle of convergence. It is shown that the semi-circle cannot be replaced by a larger arc. It remains an open question to find a corresponding theorem for the case in which $\{e_k\}$ is a given sequence of complex numbers of absolute value one. *V. F. Cowling.*

Matthies, Karl. Eine Bestabschätzung des Konvergenzradius der Potenzreihe der Umkehrfunktion einer analytischen Funktion. Arch. Math. 7 (1957), 457–458.

Let $f(z) = \sum_1^\infty a_n z^n$, $a_1 \neq 0$, be a power series representing a function regular in $|z| < r$. The present paper concerns itself with the observation that if $|a_\nu| \leq \alpha_\nu$ then the power series of the inverse function of $w(z) = \alpha_1 z - \sum_{\nu=2}^\infty \alpha_\nu z^\nu$ is a majorizing series, $z(w) = \sum_1^\infty \beta_\nu w^\nu$, to the inverse function, $z(f) = \sum_1^\infty p_\nu f^\nu$, of $f(z) = \sum_1^\infty a_n z^n$. This result is useful in providing bounds for the radius of convergence of $\sum p_\nu f^\nu$. The following theorem is proven. Let $w(z) = z - \sum_2^\infty \alpha_\nu z^\nu$, $\alpha_\nu \geq 0$, and $\alpha_\nu > 0$ for at least one ν. The radius of convergence R of the inverse function, $z(w) = \sum_1^\infty \beta_\nu w^\nu$, is given by $R = w(z_0)$, in which either (a) z_0 is the smallest real positive root of $w'(z)$ in $|z| < r$, or (b) $z_0 = r < \infty$ in case $w'(z)$ does not vanish in $|z| < r$. *V. F. Cowling.*

Liverman, T. P. G. Zeros of neighboring holomorphic functions. Proc. Nat. Acad. Sci. U.S.A. 43 (1957), 276–278.

Let the functions $g(z)$ and $f(z)$ be holomorphic throughout the closed disk Δ_ρ defined by $|z| \leq \rho$. Assume that the origin is a zero of multiplicity p (≥ 1) of $f(z)$ and that

$f(z)\neq 0$ for $0<|z|\leqq\rho$. Let M and m denote, respectively, the maximum and the minimum of $|f(z)|$ on the boundary of Δ_ρ and define $\|f-g\|_\rho=\max|f(z)-g(z)|$ for $z\in\Delta_\rho$. By Rouché's theorem, $\|f-g\|<m$ implies the existence of exactly p zeros of $g(z)$ in Δ_ρ, say $\zeta_1, \zeta_2, \cdots, \zeta_p$; assume that there are exactly $p-l(\leqq p)$ of these zeros which do not coincide with the origin. Then

$$\frac{\rho^{l+1}}{M+m}\left|\frac{g^{(l)}(0)}{l!}\right|\leqq|\zeta_i|\leqq\left\{\frac{\|g-f\|}{m}\right\}^{1/p}\rho \quad (\zeta_i\neq 0),$$

$$\frac{\rho^p}{M+m}\left|\frac{g^{(l)}(0)}{l!}\right|\leqq\prod_{\zeta_i\neq 0}|\zeta_i|.$$

{The inequalities stated by the author contain several misprints which have been corrected in the above statement.} These bounds are used to investigate the speed of convergence of the zeros of a sequence of functions and in particular the zeros of partial sums of Dirichlet series and Taylor series. *A. Edrei* (Syracuse, N.Y.).

Bagemihl, F. On the set of values assumed by holomorphic functions near essential singularities. Math. Z. 67 (1957), 49–50.

It is shown that if the set E of essential singularities of a holomorphic function $f(z)$ is of linear measure zero, then there exists a residual set of complex numbers, each of which is assumed by $f(z)$ in every neighborhood of each point of E. As pointed out by the author, the idea behind the proof of this theorem goes back to a method which was used by Osgood [Lehrbuch der Funktionentheorie, Bd. 1, 5. Aufl., Teubner, Leipzig-Berlin, 1928, p. 329] to complement Weierstrass' theorem on the behavior of a holomorphic function in the neighborhood of an isolated essential singularity. *W. Seidel* (Notre Dame, Ind.).

R.-Salinas, Baltasar Moments de fonctions analytiques et problème de Watson. J. Math. Pures Appl. (9) **35** (1956), 359–382.

Let $F(z)$ be analytic in an angle of opening $\pi\alpha$ and satisfy $\limsup\{\log|F(z)|\}/H(z)<\infty$. The author makes contributions to two parallel problems, the first being to find conditions under which $F\equiv 0$ if its absolute moments $\mu_n=\int_0^\infty|F(re^{i\phi})|r^n dr$ satisfy $\mu_n\leqq m_n(\phi)$; the second, to find conditions under which $F\equiv 0$ if $|F(re^{i\phi})|\leqq r^{-n}m_n(\phi)$ (Watson's problem). He solves both problems when $\int^\infty r^{-1-1/\alpha}H(r)dr$ converges and $m_n(\phi)=+\infty$ for $\phi\neq\phi_0$, and the first when $H(r)=+\infty$ and

$$m_n(\phi)=m_n\cdot\{\cos(\phi/\alpha)\}^{-p}.$$

[The second problem was solved by Carleman when $H(r)=+\infty$ and $m_n(\phi)=m_n$.] He also obtains several results in which $F\equiv 0$ if suitable conditions are imposed on μ_n and the ordinary moments $a_n=\int_0^\infty F(re^{i\phi})r^n dr$.
R. P. Boas, Jr. (Evanston, Ill.).

Flett, T. M. On a theorem of Lindelöf concerning prime ends. Tôhoku Math. J. (2) **8** (1956), 273–274.

Von Lindelöf [Acta Soc. Sci. Fenn. **46** (1915), no. 4] stammt der Satz: Bildet $w=f(z)$, $z=\psi(w)$ ein einfach zusammenhängendes, beschränktes Gebiet D konform auf $|w|<1$ ab, und strebt $\{w_n\}$ gegen w_0 ($|w_0|=1$) in einem Stolzschen Winkelraum, so sind sämtliche Häufungspunkte der Folge $\{\psi(w_n)\}$ Hauptpunkte des zu w_0 gehörigen Primendes. Tsuji [Jap. J. Math. **7** (1930), 91–99] gab hierfür einen neuen durchsichtigen Beweis über zwei Hilfssätze, von denen der zweite hier neu beweisen wird. Dieser lautet: Bezeichnet T_n den Kreisringsektor $\frac{1}{2}\rho_n<|1-w|<\rho_n$, $|w|<1$, $|\arg(1-w)|<\frac{1}{2}\pi-\delta$ ($\delta>0$) für eine Folge $\{\rho_n\}$ mit $\rho_{n+1}\leqq\frac{1}{2}\rho_n<1$, so gibt es eine Teilfolge $\{n_k\}$ derart, daß die Bilder von T_{n_k} unter der Abbildung $z=\psi(w)$ gegen einen Punkt des Randes von D konvergieren. Der neue Beweis fußt auf dem Auswahlsatz von Montel: Statt die Funktion $\psi(w)$ am Rande von $|w|<1$ zu untersuchen, wird die Funktionenfolge $\{\psi_n(w)\}$ mit

$$\psi_n(w)=\psi(1+\rho_n\rho_0^{-1}(w-1))$$

(Druckfehler!) in einem abgeschlossenen Teil von $|w|<1$ untersucht, ein Beweisprinzip, das von Montel stammt [vgl. auch Walsh und Gaier, Arch. Math. **6** (1954), 77–86, und die dort zitierte Literatur; MR **16**, 348]. *D. Gaier.*

Rahman, Qazi Ibadur. A note on entire functions defined by Dirichlet series. Math. Student **24** (1956) 203–207 (1957).

Let $f(s)$ be an entire function of lower order λ defined in some half plane by the Dirichlet series $\sum a_n e^{s\lambda_n}$, where $\lambda_1\geqq 0$, $\lambda_n\uparrow\infty$, $s=\sigma+it$. Let $\mu(\sigma)$ be the maximum of $|a_n|e^{\sigma\lambda_n}$, $\lambda_{N(\sigma)}$ the λ_n corresponding to this maximum term; let $M^n(\sigma)$ be the least upper bound of $|f^{(n)}(s)|$ for $\Re(s)=\sigma$ (in the half plane of absolute convergence). The author proves that if $\limsup\lambda_n^{-1}\log n=0$ then

$$\lambda=\liminf\sigma^{-1}\log\log\mu(\sigma)=\liminf\sigma^{-1}\log\lambda_{N(\sigma)};$$

and that if $a_n\geqq 0$ then for $\xi>\sigma$ the numbers $M^n(\sigma)/M^n(\xi)$ decrease as n increases. *R. P. Boas, Jr.*

Srivastav, R. P. On the derivatives of integral functions. Ganita **7** (1956), 29–44.

The author proves numerous theorems about the maximum term, maximum modulus and maximum real part for an entire function and its derivatives. Representative results are as follows. Let $f(z)$ be an entire function of positive finite order ρ, lower order λ; let $\mu(r)$ be the maximum term in the power series of f and $\nu(r)$ its rank. Then

$$\limsup\log\mu(r)/\{\nu(r)\log r\}\leqq 1-\lambda/\rho,$$
$$\limsup\log\mu(r)/\{\nu(r)\log\nu(r)\}\leqq 1/\lambda-1/\rho.$$

Let $\mathscr{M}'(r)$ be the derivative of the maximum modulus $M(r)$, $M^{(1)}(r)$ the maximum modulus of f'. Then

$$M(r)r^{(\lambda-\varepsilon-1)}<\mathscr{M}'(r)<M(r)r^{(\rho+\varepsilon-1)}$$

for almost all large r; $M^{(1)}(r)\geqq\mathscr{M}'(r)$ whenever the latter exists; $\mathscr{M}'(r)>M(r)\log M(r)/(1+\varepsilon)r\log r$ for almost all large r; $\mathscr{M}'(r)\leqq M(r)$ if $\rho\leqq 1$ and $\lambda<1$. If $\lambda>1$ and subscripts refer to successive derivatives of f, then $\mathscr{M}'_s(r)$ increases with s for large r. Let $A^{(s)}(r)$ be the maximum real part of $f^{(s)}(z)$. Then $A(r)r^{\lambda-\varepsilon-1}<A^{(s)}(r)<A(r)r^{(\rho+\varepsilon-1)s}$, $\lim r^p A^{(k)}(r)/A(r)=0$ if $p<k(1-\rho)$ and $\rho<1$; if $\lambda>1$ and $p<k(\lambda-1)$, $\lim A^{(k)}(r)/\{r^p A(r)\}=\infty$. *R. P. Boas, Jr.*

Hiong, King-Lai. Sur la limitation de $T(r, f)$ sans intervention des pôles. Bull. Sci. Math. (2) **80** (1956), 175–190.

With the help of Nevanlinna's main theorems for meromorphic functions, relations between the distribution of values of a meromorphic function and its derivatives are established. Of the applications of the general theorems obtained, the following illustrates the nature of the problems treated in the paper. Let $f(x)$ be meromorphic for $x\neq\infty$, and let $a\neq\infty$, $b\neq c$, b, $c\neq 0$. Then

$$\delta(a)+(k+1)(\delta^k(b)+\delta^k(c))\leqq 2(k+1),$$

where δ and δ^k denote the deficiencies of $f(x)$ and its kth derivative $f^{(k)}(x)$, respectively. In particular, if $f^{(k)}(x)\neq b,c$, then $\delta(a)=0$. *O. Lehto* (Helsinki).

Hiong, King-Lai. Sur un théorème fondamental de M. Milloux et ses extensions. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 386–396, 397–402.

Let $f(x)$ be meromorphic in a domain D containing the origin, and let $\alpha_i(x)$ ($i=0, 1, \cdots, l$) be $(l+1)$ functions holomorphic in D, with $\alpha_l(x)\neq 0$. Let $f_l=\sum_{i=0}^{l}\alpha_i f^{(i)}$. It was shown by Milloux [Les fonctions méromorphes et leurs dérivées, Hermann, Paris, 1940; J. Math. Pures Appl. (9) **19** (1940), 197–210; MR **7**, 427; **2**, 356] that

$$(1)\quad T(r,f)<(l+1)N(r,f)+N\Big(r,\frac{1}{f}\Big)+N\Big(r,\frac{1}{f_l-1}\Big)+S(r).$$

The author has shown in the paper reviewed above that for the case $\alpha_0=0$

$$(2)\quad T(r,f)<N(r,f)+N\Big(r,\frac{1}{f-a}\Big)+N\Big(r,\frac{1}{f_l-b}\Big)-N_1(r,f)+S(r,f).$$

When $\alpha_0\neq 0$, however, the method used in obtaining (2) proceeds only in the case $a=0$ and $b=1$.

In the present paper, the author, by a different method, has succeeded in establishing (2) without restriction and has extended the theorem to the case where the constants a and b are replaced by any two holomorphic functions. A second extension is also obtained when the functions α_i are permitted to be meromorphic of order inferior to that of f. *M. S. Robertson* (New Brunswick, N.J.).

Tanaka, Chuji. On the class H_p of functions analytic in the unit circle. Yokohama Math. J. **4** (1956), 47–53.

Let $f(z)$ be regular in $|z|<1$.

$$m_p(r)=m_p(f,r)=\int_0^{2\pi}|f(re^{i\theta})|^p d\theta$$

($p>0$, $0\leqq r<1$) is an increasing function of r [Hardy, Proc. London Math. Soc. (2) **14** (1915), 269–277]. $f(z)$ belongs to the class H_p if $\lim_{r\to 1} m_p(r)=m_p<\infty$. If $f(z)\in H_p$, the radial limit $\lim_{r\to 1} f(re^{i\theta})$ exists for almost all θ [F. Riesz, Math. Z. **18** (1923), 87–95]. The author proves: If $f(z)\in H_p$, then for almost all θ there exists a simple closed Jordan curve C_θ lying in $|z|<1$ and having a common tangent with $|z|=1$ at $z=e^{i\theta}$ such that

$$\lim_{D_\theta\ni z\to e^{i\theta}} f(z)=f(e^{i\theta})$$

exists, where D_θ is the closed domain surrounded by C_θ. The following extension of a theorem by Lindelöf is also proved: If $f(z)\in H_p$, $\lim_{\varphi\to\theta+0} f(e^{i\varphi})=\alpha$, $\lim_{\varphi\to\theta-0} f(e^{i\varphi})=\beta$, then $\alpha=\beta$ and $f(z)$ tends uniformly to α as $z\to e^{i\theta}$ in the unit circle. *B. A. Amira* (Jerusalem).

Artemiadis, Nicolas K. Généralisation d'un théorème de M. S. Mandelbrojt. C. R. Acad. Sci. Paris **244** (1957), 834–836.

$\mathcal{T}(p)$ désigne la classe des fonctions $F(z)=\sum_{1,\infty} C_n^{(p)}z^n$ typiquement réelles d'ordre p (entier positif) dans le cercle $|z|<1$. Le cas $p=1$ correspond aux fonctions $F(z)=z+\sum_{2,\infty} a_n z^n$ typiquement réelles dans le cercle $|z|<1$. Supposons que $F\in\mathcal{T}(2)$ et soient 0, $\theta_1^{(r)}$, π, $2\pi-\theta_1^{(r)}$ les valeurs de θ ($z=re^{i\theta}$) pour lesquelles $\mathcal{I}(F(z))$ change de signe sur $|z|=r$. Une série $\sum a_n$ est dite $\mathfrak{M}$-sommable, s'il existe un $\lambda\geqq 0$ tel que, si l'on pose $s_i=\sum_{\nu=1}^{i} a_\nu$, la limite

$$\lim_{n=\infty}\Big[\frac{s_1+s_2+\cdots+s_{n-1}+\frac{1}{2}s_n}{n}+\frac{\lambda(s_{n+1}-s_{n-1})}{n}\Big]$$

existe, est finie et différente de zéro. En généralisant un théorème de Mandelbrojt [Bull. Sci. Math. (2) **58** (1934), 185–200], l'auteur démontre le théorème suivant: Si $F\in\mathcal{T}(2)$, si $\lim_{r=1-0} F(r)=F(1)$ existe, est finie et différente de zéro, s'il existe au moins un point d'accumulation $e^{i\theta}$ des points $re^{i\theta_1^{(r)}}$ ($r\to 1$) pour lequel $\cos\theta=\gamma\neq 1$ et $2F(1)(1-\gamma)-C_1^{(2)}\neq 0$, alors la série $\sum C_n^{(2)}$ est $\mathfrak{M}$-sommable et sa somme est égale à $F(1)$. Le théorème se généralise pour p quelconque. *S. Mandelbrojt* (Paris).

Hiong, King-Lai. Un cycle simple dans la théorie des familles normales. C. R. Acad. Sci. Paris **244** (1957), 1440–1443.

Hiong, King-Lai. Sur les fonctions holomorphes sans zéros dont les dérivées admettent une valeur exceptionnelle. C. R. Acad. Sci. Paris **244** (1957), 2125–2127.

The author continues his long series of papers on the normality of families of functions which have exceptional values. He announces the result that every family F_k of holomorphic functions in the unit disc which has properties (i)–(iii) is normal: (i) $f(z)\neq 0$ and (ii) $f^{(k)}(0)\neq 1$ for every f in F_k; (iii) the kth derivatives of the functions of F_k have 1 as a uniformly B-exceptional value. [Cf. Hiong, Ann. Sci. Ecole Norm. Sup. (3) **72** (1955), 165–197; MR **17**, 600.] The first paper deals with the case $k=0$ of this result. *J. Korevaar* (Madison, Wis.).

Eweida, M. T. A note on Abel's polynomials. Proc. Math. Phys. Soc. Egypt **5** (1955), no. 3, 63–66 (1957).

The Abel expansion of a function $f(z+\alpha)$ is given by

$$(*)\qquad f(\alpha)+\sum_{n=1}^{\infty} f^{(n)}(\alpha+n\beta)\frac{z(z-n\beta)^{n-1}}{n!}.$$

The author proves that if $f(z)$ is an integral function such that

$$\limsup_{r\to\infty}\frac{\log M(r)}{r}<\frac{\tau}{\beta},$$

where $\tau=.278\cdots$ is the unique positive root of $xe^{1+x}=1$, then the series (*) (with $\alpha=0$) converges absolutely and uniformly to $f(z)$ in any finite region of the plane.

The author claims to correct an error of Watson concerning the convergence of this expansion [J. London Math. Soc. **3** (1928), 188–192]. However, the author confuses a criterion for the uniqueness of the expansion (*) with one for the convergence of the series. A correct result on convergence is given in Watson's paper. *M. Newman* (Washington, D.C.).

Wintner, Aurel. On the local domains of regularity of functions defined by implicit conditions. Rend. Circ. Mat. Palermo (2) **5** (1956), 275–287 (1957).

Let $f(z,w)$ be a regular complex function of two complex variables z, w, in a neighborhood $U\equiv(|z|<a, |w|<b)$ of $(z,w)=(0,0)$. The author considers the parallelism between the problems of the existence of regular solutions for the equations $dw/dz=f(z,w)$ and $w=zf(z,w)$. The main result (principle of subordination) is as follows: If there exists a real function $\varphi(r,s)$ of real variables r, s, which is continuous for $0\leqq r<a$ and $0\leqq s<b$, and such that (A) $|f(z,w)|\leqq\varphi(|z|,|w|)$ on U, and (B) on a certain

interval $0 \leq r < \alpha$, with $0 < \alpha \leq a$, the equation $s = r\varphi(r, s)$ has a continuous solution $s = s(r) < b$, then the equation $w = zf(z, w)$ has a unique regular solution $w = w(z)$ with $w(0) = 0$, and $w(z) \leq s(|z|)$ on $|z| < \alpha$. The proof is derived, by analogy, from previous results of the author concerning the equation $dw/dz = f(z, w)$ [Ann. Mat. Pura Appl. (4) **41** (1956), 343–357; MR **18**, 473]. Several corollaries improving classical results are obtained and it is proved that the "principle of subordination" enlarges the domain of regularity obtained by Cauchy's "méthode des limites" for the solution of $w = zf^*(z, w)$, where

$$f^*(z, w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |c_{mn}| z^m w^n$$

is the "best majorant" of $f(z, w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{mn} z^m w^n$. The last part of the paper deals with local inverses of regular function elements, and contains a discussion of some results from two papers of Landau [S.-B. Preuss. Akad. Wiss. **1926**, 467–474; Math. Z. **30** (1929), 608–634].

A. G. Azpeitia (Providence, R.I.).

See also: Korányi, p. 126; Gehring, p. 131; van der Corput, p. 136; Haplanov, p. 136; Džrbašyan, p. 138; Koehler, p. 158; Behnke, p. 170; Remmert, p. 170; Dolbeault, p. 171.

Geometric Analysis

Shapiro, Victor L. The divergence theorem without differentiability conditions. Proc. Nat. Acad. Sci. U. S. A. 43 (1957), 411–412.

Let X denote the point (x, y) in the plane and let $V(X) = (A(X), B(X))$ be a continuous vector field defined in a neighborhood of X_0. The divergence of V at X_0 is defined by

$$\operatorname{div} V(X_0) = \lim_{t \to 0} (\pi t^2)^{-1} \int_{C(X_0, t)} (A dy - B dx)$$

if this limit exists as a finite number; here $C(X_0, t)$ is the circle with center at X_0 and radius t. The formula

$$\int_C (A(X) dy - B(X) dx) = \iint_D \operatorname{div} V(X) dx dy$$

is proved under the following conditions: D is a bounded domain, its boundary C is the union of a finite number of simple closed rectifiable curves, $V(X)$ is defined and continuous on the closure $\bar{D}$ of D, div $V(X)$ exists at all points of D except possibly on a closed subset of $\bar{D}$ whose logarithmic capacity is zero, and div $V(X)$ is Lebesgue integrable on D.

W. Rudin (Rochester, N.Y.).

See also: Andrade Guimarães, p. 113; Auslander, p. 147; Lelong-Ferrand, p. 168.

Harmonic Functions, Convex Functions

Elkins, Thomas A. Orthogonal harmonic functions in space. Proc. Amer. Math. Soc. 8 (1957), 500–509.

Two harmonic functions u, v in a region of space are called orthogonal if $\nabla u \cdot \nabla v = 0$ in the region (equivalently, if uv is again harmonic there). Given a harmonic function u, will there be a non-constant harmonic function orthogonal to it? If u depends only on 1 or 2 variables, the answer is yes, but a counter-example is given to the general case. It is the polynomial $x^3 + y^3 - 3xz^2 - 3yz^2$. No second degree polynomial can serve as a counter-example.

A. B. Novikoff (Berkeley, Calif.).

Berstein, I. A topological characterization of the pseudo-conjugate of a pseudoharmonic function. Rev. Math. Pures Appl. 1 (1956), no. 1, 45–48.

A translation from the Romanian of the article reviewed in MR 17, 145.

Ivanov, V. K. On the distribution of singularities of a potential. Uspehi Mat. Nauk (N.S.) 11 (1956), no. 5(71), 67–70. (Russian)

Two formulae are deduced in this paper for the depth H of the set of singularities of a potential V for which $g = \partial V / \partial z$ is known on the surface of the ground, $z = 0$. The first deals with the two-dimensional case when the isograms are straight lines parallel to OY, so that $g = g(x)$ is known on a profile $-\infty < x < +\infty$, $y = \text{constant}$. The result is $H = -\limsup_{r \to \infty} \{r^{-1} \cdot \ln |v(r)|\}$, where $v(r)$ is the Fourier transform of $g(x)$ in the interval $(-\infty, \infty)$. The second concerns the general case when $g = g(x, y)$ is known all over the plane XOY, $z = 0$, and then

$$H = \inf_{0 \leq \varphi \leq 2\pi} \left[-\limsup_{r = \infty} \{r^{-1} \cdot \ln w(r, \varphi)\} \right]$$

where again

$$w(r, \varphi) = (2\pi)^{-1} \int_0^{\infty} \int_0^{2\pi} e^{ir\rho \cos(\theta - \varphi)} g(\rho, \theta) \rho d\rho d\theta,$$

ρ and θ being polar coordinates in the plane $z = 0$. The practical value of these formulae is doubtful because of the necessity of extending the integrations very far from the origin in order to obtain an estimate more or less correct of $v(r)$ or of $w(r, \varphi)$.

E. Kogbetliantz.

Gehring, F. W. On the radial order of subharmonic functions. J. Math. Soc. Japan 9 (1957), 77–79.

The author generalizes a theorem due to Seidel and Walsh [Trans. Amer. Math. Soc. **52** (1942), 128–216; MR **4**, 215] by proving the following result: If $w(z) \geq 0$ is subharmonic in $|z| < 1$ and if, for some $p > 1$,

$$\iint_{|z| < 1} w^p(z) dx dy < \infty,$$

where $z = x + iy$, then, for almost all θ in $-\pi \leq \theta < \pi$, $w(z) = o\{(1 - |z|)^{-1/p}\}$ uniformly as $z \to e^{i\theta}$ in each Stolz domain. The proof makes use of a maximal theorem of Hardy and Littlewood [Acta Math. **54** (1930), 81–116]. The above result also sharpens a recent result due to Tsuji [J. Math. Soc. Japan **6** (1954), 336–342; MR **16**, 809].

W. Seidel (Notre Dame, Ind.).

Arsove, Maynard G. Mass distributions for products of subharmonic functions. Duke Math. J. 24 (1957), 215–225.

If u_1, u_2, and $u_1 u_2$ are continuous subharmonic functions in a plane domain Ω, then the reviewer [Bull. Amer. Math. Soc. **53** (1947), 89–95; MR **8**, 461] and Ishikawa [Proc. Japan. Acad. **30** (1954), 532–537; MR **16**, 588], proved that for all Borel sets E, such that $\bar{E} \subset \Omega$, we have

$$(1) \quad m(E) = \int_E u_1 dm_2 + \int_E u_2 dm_1 - \frac{1}{\pi} \int_E \left(\frac{\partial u_1}{\partial x} \frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y} \frac{\partial u_2}{\partial y} \right) da,$$

where m_1, m_2 and m are the respective mass distributions, and da is the element of area. Drawing on recent results of Brelot [Ann. Inst. Fourier, Grenoble **5** (1953–1954), 371–419; MR **17**, 603] and Deny [Acta Math. **82** (1950), 107–183; MR **12**, 98], the author establishes (1) for arbitrary functions u_1 and u_2 in the algebra $\mathscr{A}$ of all functions representable as the differences of locally bounded functions subharmonic in Ω. The author then proceeds to use his generalization of (1) to obtain several interesting results. We quote two of them. (I) Let u be a function in $\mathscr{A}$, and let φ be a real-valued function having a Lipschitzian derivative on an interval containing the range of u. Then it follows from an earlier result due to the author [Trans. Amer. Math. Soc. **75** (1953), 327–365; MR **15**, 526] that the composite function $\varphi\circ u$ is almost δ-subharmonic in Ω. If μ and m are the mass distributions associated with $\varphi\circ u$ and u, then

$$\mu(E)=\int_E \varphi'\circ u\, dm-\frac{1}{2\pi}\int_E\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]\varphi''\circ w\, da.$$

(II) If u_1 and u_2 are in $\mathscr{A}$, and if the mass distributions m_1, m_2, associated with u_1 and u_2, are absolutely continuous, then

$$\lim_{r\to 0}\frac{1}{\pi r^2}\int_0^{2\pi}[u_1(z+re^{i\theta})-u_1(z)][u_2(z+re^{i\theta})-u_2(z)]d\theta=\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial x}+\frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial y}.$$

M. O. Reade (Ann Arbor, Mich.).

See also: Shapiro, p. 131; Bertram, p. 181.

Special Functions

Carlitz, L. Some arithmetic properties of the Legendre polynomials. Proc. Cambridge Philos. Soc. **53** (1957), 265–268.

Cette étude très courte, mais extrèmement riche de résultats variés, tous relatifs à un domaine à peu près inexploré, déduit de la formule de Good

$$P_n(x)=\frac{1}{t}\sum_{r=0}^{t-1}\left\{x+\sqrt{(x^2-1)}\cos\frac{2\pi r}{t}\right\}^n\quad(t>n)$$

(P_n désignant les polynomes de Legendre) des propriétés arithmétiques de $P_n(x)$. Par example, si on pose

$$W_m(x)=\sum_0^m\binom{m}{r}^2x^r,$$

on montre que, si $p\equiv 1 \pmod 4$,

$$P_m(3)\equiv W_m(-1)\equiv 2a \pmod p,$$

si a est l'unique entier impair tel que $p=a^2+b^2$ et $a\equiv b+1 \pmod 4$.

D'autres propriétés sont obtenues à partir de la formule d'addition des polynomes de Legendre. *R. Campbell.*

Al-Salam, W. A. On the product of two Legendre polynomials. Math. Scand. **4** (1956), 239–242.

On sait que Neumann et Adams ont résolu le problème, devenu classique, de la transformation d'un produit de deux polynomes de Legendre en une forme linéaire de polynomes de Legendre

$$P_p(x)P_q(x)=\sum_0^q\lambda_r P_{p+q-2r}(x)\quad(p\geq q).$$

Bailey a généralisé les formules au cas des fonctions de Legendre associées. L'auteur résoud ici le problème inverse: transformer un polynome de Legendre $P_{p+q}(x)$ en une forme linéaire de produits de tels polynomes

$$\sum_{k=0}^q A_kP_{p-k}(x)P_{q-k}(x),$$

et étend sa méthode aux fonctions associées $P_n^m(x)$.

R. Campbell (Caen).

Lakshmana Rao, S. K. Characteristic relations for the ultraspherical polynomials. Proc. Indian Acad. Sci. Sect. A. **45** (1957), 172–176.

Quand on cherche à étendre aux polynomes de Gegenbauer l'inégalité de Turán relative aux polynomes de Legendre, on tombe sur diverses relations concernant les $P_n^\gamma(x)$.

L'objet de la présente étude est de montrer que ces relations caractérisent précisément les polynomes ultrasphériques. Ainsi si $\Delta_n^\gamma(x)$ désigne

$$[P_n^\gamma(x)]^2-P_{n+1}^\gamma(x)P_{n-1}^\gamma(x),$$

la relation

$$\frac{d}{dx}\Delta_n^\gamma(x)=\frac{1-\gamma}{n+\gamma}W[P_{n+1}(x),P_{n-1}(x)]$$

(W désignant le wronskien) caractérise les $P_n^\gamma(x)$.

Il en est ainsi encore pour une équation aux différences finies à laquelle satisfait $d\Delta_n^\gamma(x)/dx$.

R. Campbell (Caen).

Ghurye, S. G. A characterization of the exponential function. Amer. Math. Monthly **64** (1957), 255–257.

Let $f(x)$ be real-valued and continuous for $x\geq 0$. For each x and each positive integer n, let $k_n(x)$ be the number of distinct points in the set $f(x)$, $\{f(x/2)\}^2, \cdots, \{f(x/n)\}^n$. Theorem: If $k_n(x)\leq k$ (independent of x and n), and if $f(x)>0$ for some $x\geq 0$, then a real number α exists such that $f(x)=e^{\alpha x}$ (for all $x\geq 0$).

If $f(x)$ is complex-valued, continuous, and $k_n(x)\leq k$, a conjecture is made concerning the form of $f(x)$.

I. M. Sheffer (University Park, Pa.).

Mikolás, Miklós. Zur Theorie der Gammafunktion, der Riemannschen Zetafunktion und verwandter Funktionen. I. Acta Math. Acad. Sci. Hungar. **6** (1955), 381–438. (Russian summary)

Ausgehend von der summatorischen Funktion

$$\beta_p(z,n)=\sum_{k=0}^{n-1}(z+k)^p\ (p \text{ ganz})$$

werden vom Verf. die klassischen Eigenschaften der Funktion

$$\psi(z)=\lim_{n\to\infty}[\log n-\beta_{-1}(z,n)]=\frac{\Gamma'(z)}{\Gamma(z)}$$

entwickelt. Durch Integration entspringen die Ergebnisse für $\Gamma(z)$, bzw. $\log\Gamma(z)$. Durch die sehr leicht sich ergebende Erzeugungsfunktion

$$e^{zw}\frac{e^{nw}-1}{e^w-1}=\sum_{p=0}^\infty\beta_p(z,n)\frac{w^p}{p!}$$

werden die Bernoullischen Polynome als Hilfsfunktionen durch

$$\frac{we^{zw}}{e^w-1}=\sum_{k=0}^\infty B_k(z)w^k$$

eingeführt. Dieser Weg erlaubt die auftretenden Reihen und Produktentwicklungen mit Restglieder zu versehen. Es leuchtet durchwegs unschwer ein, welcher Wert des (unendlichvieldeutigen) komplexen logarithmus, bzw. welcher Zweig von log $\Gamma(z)$, zu verstehen ist. Die Arbeit wird beschlossen mit einem ausgedehnten Literaturverzeichnis. *S. C. van Veen* (Delft).

Janković, Zlatko. Note on spherical Bessel functions. Glasnik Mat.-Fiz. Astr. Ser. II. **11** (1956), 143-148. (Serbo-Croatian summary)

Recherchant le développement de $e^{ikr\cos\theta}$ sous la forme, d'ailleurs classique, $\sum_{n=0}^{\infty} a_n f_n(r) P_n(\cos\theta)$, l'auteur retrouve le développement en série entière de $f_n(r)$ (fonction de Bessel sphérique). *R. Campbell* (Caen).

Hochstadt, Harry. A relationship between parabolic and spherical functions. Proc. Amer. Math. Soc. **8** (1957), 489–491.

Envisageant une formule classique de Gegenbauer (théorème d'addition):

$$\frac{J_{\nu-\frac{1}{2}}(r\sin\theta\sin\theta')}{(r\sin\theta\sin\theta')^{\nu+\frac{1}{2}}}e^{ir\cos\theta\cos\theta'}=$$

$$\frac{2^{2\nu}\Gamma^2(\nu)}{(2\pi)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{i^m m!(m+\nu)}{\Gamma(m+2\nu)}\frac{J_{\nu+m}}{r^\nu}C_m^\nu(\cos\theta)C_m^\nu(\cos\theta')$$

l'auteur remarque que le changement de paramètre $\operatorname{tg}\frac{1}{2}\theta'=\sqrt{t}$ transforme le membre de gauche de cette équation en la fonction génératrice des fonctions paraboliques. (Il importe de préciser que l'auteur appelle ainsi non les fonctions de Weber, mais celles du paraboloide de révolution.) Développant alors le 1er membre en série entière de t, il obtient une relation (à une infinité de termes) entre ces fonctions et celles de Gegenbauer. *R. Campbell* (Caen).

Altmann, S. L. On the symmetries of spherical harmonics. Proc. Cambridge Philos. Soc. **53** (1957), 343–367.

The purpose of this paper is to obtain expansions in spherical harmonics that belong to a given irreducible representation of a symmetry group. The method used here gives the desired expansions for the spherical harmonics of all orders, both for the cyclic and the dihedral groups (but not for the cubic groups). This method is a straightforward application of a standard technique in group theory by which the problem reduces to finding the transforms $\mathscr{R}Y_l^m$ of the spherical harmonics Y_l^m under the operations $\mathscr{R}$ of a group. It is first suposed that $\mathscr{R}$ is a pure rotation; we have

$$\mathscr{R}Y_l^m=\sum_{m'}Y_l^{m'}D^{(l)}(\mathscr{R})_{m'm}.$$

So starting from the normalized spherical harmonics

$$Y_l^m(\theta,\varphi)=\left[\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}\right]^{1/2}P_l^m(\cos\theta)e^{im\varphi},$$

and specifying a rotation by the Euler angles α, β, γ, the author furnishes the elements of the matrix $D^{(l)}(\mathscr{R})$.

Some particular cases are examined. The extension of these results for a general space-group is simple. Numerical tables are given; the case of the tetrahedral group T is considered in more detail. Then tables are given of spherical harmonics for all the crystallographic point groups (cyclic, dihedral and cubic groups). An example of the use of the tables is explained. The notation is simple and facilitates applications (especially chemical applications. *R. Campbell* (Caen).

Dingle, R. B. The Fermi-Dirac integrals

$$\mathscr{F}_p(\eta)=(p!)^{-1}\int_0^\infty \epsilon^p(e^{\epsilon-\eta}+1)^{-1}d\epsilon.$$

Appl. Sci. Res. B. **6** (1957), 225–239.

Dingle, R. B. The Bose-Einstein integrals

$$\mathscr{B}_p(\eta)=(p!)^{-1}\int_0^\infty \epsilon^p(e^{\epsilon-\eta}-1)^{-1}d\epsilon.$$

Appl. Sci. Res. B. **6** (1957), 240–244.

Dingle, R. B.; Arndt, Doreen; and Roy, S. K. The integrals

$$\mathfrak{E}_p(x)=(p!)^{-1}\int_0^\infty \epsilon^p(1+x\epsilon^3)^{-1}e^{-\epsilon}d\epsilon$$

and

$$\mathfrak{F}_p(x)=(p!)^{-1}\int_0^\infty \epsilon^p(1+x\epsilon^3)^{-2}e^{-\epsilon}d\epsilon$$

and their tabulation. Appl. Sci. Res. B. **6** (1957), 245–252.

The author discusses the principal properties of the special functions described in the titles. These functions are of interest in physics particularly in the theory of semi-conductors. Series expansions, asymptotic expansions, and relationships with other special functions are treated. A list is given of the available tables of these functions and the following new tables are given: $\mathscr{F}_p(\eta)$ for $p=0$, 1, 2, 3, 4 and $0\leqq\eta\leqq10$, also $p=-1$, 0 and $-4\leqq\eta\leqq0$; $\mathfrak{E}_p(x)$ for $p=0, \frac{1}{2}, 1, \frac{3}{2}, \cdots, \frac{13}{2}$ and $0\leqq x\leqq20$; $\mathfrak{F}_p(x)$ for $p=0, \frac{1}{2}, 1, \frac{3}{2}, \cdots, 10$ and $0\leqq x\leqq20$. *I. I. Hirschman, Jr.* (St. Louis, Mo.).

Kazarinoff, Nicholas D. Asymptotic forms for the Whittaker functions with both parameters large. J. Math. Mech. **6** (1957), 341–360.

Le comportement asymptotique des solutions de l'équation

$$\text{(1)}\qquad \frac{d^2W}{dx^2}+\left\{-\frac{1}{4}+\frac{k}{x}+\frac{(1/4)-m^2}{x^2}\right\}W=0$$

a été peu étudie dans le cas où les deux paramètres k et m sont grands. L'objet de cette étude est de déterminer ce comportement si x, k, m sont complexes, $|k|$ et $|m|$ grands, et la quantité $\sigma=(m^2-k^2)/k$ est bornée et telle que $\lim_{|k|\to\infty}(m/k)=\pm1$. Si $|x/2k|\geqq e^\beta$ $(\beta>0)$, $0\leqq\arg x\leqq\pi$, $|\arg k|\leqq\pi$, $|k|>N$ et si $|\sigma|<M$, alors

$$W_{k,m}(x)=\left(1-\frac{2k}{x}\right)^{-\frac{1}{2}(\sigma+1)}e^{-\frac{1}{2}x}x^k\left[1+\frac{E(x,k)}{k}\right],$$

$$W_{-k,m}(e^{-\pi i}x)=\left(1-\frac{2k}{x}\right)^{-\frac{1}{2}(\sigma-1)}e^{k\pi i+\frac{1}{2}x}x^{-k}\left[1+\frac{E(x,k)}{k}\right],$$

$E(x, k)$ désignant des fonctions uniformément bornées en x et k, si β et N sont assez grands.

Il y a une forme très importante de l'équation (1) pour les applications à la théorie de MacKelvey, c'est celle obtenue en posant

$$z=\log(x/2k)\quad\text{et}\quad u=e^{-\frac{1}{2}z}W,$$

qui s'écrit

$$\text{(2)}\qquad \frac{d^2u}{dz^2}-\{k^2(e^z-1)^2+k\sigma\}u=0.$$

Quand z appartient à un certain domaine, le comportement asymptotique d'une telle équation est obtenue en reliant les solutions de (2) aux fonctions de Weber. Tous les résultats sont facilement transposables aux polynomes de Laguerre L_n^α dont ils fournissent la forme asympto-

tique lorsque n est bornée et $|\alpha|\to\infty$, la quantité

$$-\sigma=\frac{n^3+(\alpha+1)n+\frac{1}{2}(2\alpha+1)}{n+\frac{1}{2}(\alpha+1)}$$

restant bornée. *R. Campbell* (Caen).

See also: Halmos, p. 112; Galler, p. 113; Copeland, p. 113: Eichler, p. 123; Heflinger, p. 136; Salzer, p. 136; Kuz'mina, p. 137; Feldmann, p. 141; Vilenkin, p. 153; Rosati, p. 173; Zmuda, p. 174; Penndorf and Goldberg, p. 181; Dekanosidze, p. 181; Head and Wilson, p. 182; Fisz, p. 184.

Sequences, Series, Summability

Kosmák, Ladislav. On certain sequences of sets of points on a circle. Časopis Pěst. Mat. **80** (1955), 299–307. (Czech)

From a set $S^{(1)}=\{A_0^{(1)}, A_1^{(1)}, \cdots, A_{n-1}^{(1)}\}$ of n points arranged, in this order, on a circle, a sequence of similar sets $S^{(k)}=\{A_0^{(k)}, A_1^{(k)}, \cdots, A_{n-1}^{(k)}\}$ is defined recursively as follows: the ith point $A_i^{(k+1)}$ of $S^{(k+1)}$ divides the arc $A_i^{(k)}A_{i+1}^{(k)}$ in the ratio b: $(1-b)$, where $0<b<1$ $(A_n^{(k)}=A_0^{(k)})$. The author proves: (i) if $a_i^{(k)}$ is the angle subtended by $A_i^{(k)}A_{i+1}^{(k)}$ at the centre of the circle, then $\lim_{k\to\infty} a_i^{(k)}=2\pi/n$; (ii) if b is a rational number p/q (p, q relatively prime), then the $A_i^{(k)}$ have exactly nq limit points, equally spaced round the circle; and (iii) if b is irrational, then every point on the circle is a limit point. This generalizes a result by J. Brejcha who considered the case $n=4$, $b=\frac{1}{2}$ [Prace Moravskoslezské Akad. Ved. Prirod., Brno **24** (1952), 347–358; MR **16**, 277].

F. A. Behrend (Melbourne).

Hájek, Jaroslav. Remark on the article "On certain sequences of sets of points on a circle". Časopis Pěst. Mat. **81** (1956), 77–78. (Czech)

The author gives a shorter proof of the result (i) of the paper reviewed above by showing: if $a_i^{(k)}$ $(i=0, \cdots, n-1)$ are n sequences of real numbers defined recursively by

$$a_i^{(k+1)}=(1-b)a_i^{(k)}+ba_{i+1}^{(k)},$$

where $0<b<1$ and $a_n^{(k)}=a_0^{(k)}$, then

$$\lim_{k\to\infty} a_i^{(k)}=\sum_{m=0}^{n-1} a_m^{(1)}/n.$$

The author also observes that the result follows directly from the theory of Markov chains. *F. A. Behrend.*

Blambert, M. Quelques propriétés de répartition des singularités d'une série de Dirichlet générale en relation avec la nature de la suite des coefficients. Publ. Sci. Univ. Alger. Sér. A. **2** (1955), 251–272 (1957).

En utilisant les méthodes de son Mémoire antérieur [Ann. Sci. Ecole Norm. Sup. (3) **72** (1955), 199–235; MR **17**, 722], l'auteur indique une nouvelle démonstration du théorème de Cramér-Pólya-Bernstein, et en donne quelques applications à la théorie générale des singularités d'une série de Dirichlet. *S. Mandelbrojt* (Paris).

Alexiewicz, A. On Cauchy's condensation theorem. Studia Math. **16** (1957), 80–85.

Let $\{\varepsilon_n\}$ be a non-increasing sequence of non-negative numbers and let $\{a_n\}$ be a sequence of non-negative numbers. It is shown that a necessary and sufficient condition that the series

$$\sum_{n=1}^{\infty} a_n\varepsilon_n \text{ and } \sum_{n=1}^{\infty} \varepsilon_n$$

converge or diverge simultaneously for every such sequence $\{\varepsilon_n\}$ is that

$$0<\liminf_{n\to\infty} \frac{a_1+\cdots+a_n}{n} \leq \limsup_{n\to\infty} \frac{a_1+\cdots+a_n}{n}<\infty.$$

Cauchy's condensation theorem is a special case of this theorem in which $a_n=n$ if $n=2^k (k=1, 2, \cdots)$ and $a_n=0$ otherwise.

The proof is obtained by considering the Banach space $\mathfrak{X}(\mathfrak{a})$ consisting of all sequences $\mathfrak{x}=\{x_n\}$ tending to zero for which there exists a non-increasing sequence $\{\varepsilon_n\}$ such that $|x_n|\leq\varepsilon_n$ and

$$\sum_{n=1}^{\infty} a_n\varepsilon_n<\infty, \quad \mathfrak{a}=\{a_n\}.$$

The norm is $\sum_{n=1}^{\infty} a_n[\sup\{|x_n|, |x_{n+1}|, \cdots\}]$. It is shown that the general form of linear functionals on this space is

$$\xi(\mathfrak{x})=\sum_{n=1}^{\infty} b_nx_n,$$

where

$$\operatorname{norm}\xi=\sup_{1,2,\cdots} \frac{|b_1|+\cdots+|b_n|}{a_1+\cdots+a_n}<\infty.$$

J. G. Herriot (Stanford, Calif.).

Syrmus, T. Series with unbounded partial sums in the summability field of a matrix method. Uč. Zap. Tartu. Gos. Univ. **42** (1956), 143–151. (Estonian. Russian summary)

The following theorem is known [Mazur and Orlicz, C. R. Acad. Sci. Paris **196** (1933), 32–34; Studia Math. **14** (1954), 129–160; MR **16**, 814; K. Zeller, Math. Z. **53** (1951), 463–487; MR **12**, 604]: (i) A conservative matrix method (*) $\lim_n \sum_k a_{nk}s_k$ (sequence-to-sequence-transformation) sums an unbounded sequence if it sums a bounded divergent sequence. One cannot immediately carry over (i) to methods (**) $\sum_n \sum_k \alpha_{nk}u_k$ (series-to-series-transformation), since not every method (**) corresponds to a method (*) [see, e.g., Rechard, Proc. Amer. Math. Soc. **2** (1951), 730–731; MR **13**, 339]. The author considers the summability field of (**) and certain subdomains of it as FK-spaces and gives representations of continuous linear functionals in these spaces. In this way he proves the truth of (i) and certain extensions for methods (**). *K. Zeller* (Tübingen).

Borwein, D. On a scale of Abel-type summability methods. Proc. Cambridge Philos. Soc. **53** (1957), 318–322.

Der Verf. untersucht eine Verallgemeinerung der Abel-Summierbarkeit: s_n heisst A_λ-summierbar zum Wert s, wenn

$$(1-x)^{\lambda+1}\sum_{n=0}^{\infty}\binom{n+\lambda}{n}s_nx^n$$

für $x\to1-0$ gegen s strebt. Für diese Verfahren gilt: (1) A_λ ist permanent für $\lambda>-1$, (2) $A_\lambda\subset A_\mu$ im engeren Sinn für $\lambda>\mu>-1$, (3) $(C, \alpha)\subset A_\lambda$ für alle $\alpha>-1$, $\lambda>-1$, (4) $A_\lambda\subset A_\lambda H_\chi$, $\lambda>-1$, für jedes Hausdorffverfahren H_χ mit absolut stetigem χ. *A. Peyerimhoff.*

Prasad, B. N.; and Pati, T. On the second theorem of consistency in the theory of absolute Riesz summability. Trans. Amer. Math. Soc. 85 (1957), 122–133.

Let λ denote a sequence $\{\lambda_n\}$ such that $0\leqq\lambda_n\uparrow\infty$, and let $r\geqq0$. A series $\sum c_n$ is said to be summable (R, λ, r) if $A_\lambda{}^r(\omega)=\sum_{\lambda_n\leq\omega}(1-\lambda_n/\omega)^r c_n$ approaches a limit as $\omega\rightarrow\infty$. The series is called summable $|R, \lambda, r|$ if $A_\lambda{}^r(\omega)$ is of bounded variation on some interval (h, ∞). The authors give a condition on $\varphi(t)$ under which summability $|R, \lambda_n, r|$ of a series implies summability $|R, \varphi(\lambda_n), r|$ for non-integral orders r. The condition is as follows: $\varphi(t)\geqq0, \rightarrow\infty$ as $t\rightarrow\infty$, $\varphi'(t)\uparrow$, $\varphi(t)$ a $(k+2)$th indefinite integral where k is the integral part of r, and $t^s\varphi^{(s)}(t)/Q(t)$ of bounded variation on some interval (h, ∞) for $s=1, 2, \cdots, k+1$. This condition is similar to one given by Pati for integral r [Quart. J. Math. Oxford Ser. (2) 5 (1954), 161–168; MR 16, 351]. *J. Korevaar* (Madison, Wis.).

Guha, U. The "Second theorem of consistency" for absolute Riesz summability. J. London Math. Soc. 31 (1956), 300–311.

The 'second theorem of consistency' for Riesz means, proved by G. H. Hardy [Proc. London Math. Soc. (2) 15 (1916), 72–88] states that if $\mu(x)$ is a logarithmico-exponential function of x, and $\mu(x)=O(x^\Delta)$, where Δ is a positive constant, however large, then $(R, \lambda, k)\sum a_n=A$ implies $(R, \varphi, k)\sum a_n=A$, where $\varphi(x)=\mu\{\lambda(x)\}$. K. A. Hirst gave a generalization of this theorem [ibid. (2) 33 (1932), 353–366] in which the condition that μ be a logarithmico-exponential function was replaced by order-conditions on μ and its derivatives. Hirst's conditions are simpler when k is integral than when it is non-integral. B. Kuttner gave a further simplification and extension of Hirst's theorem [J. London Math. Soc. 26 (1951), 104–111; 27 (1952), 207–217; MR 12, 696; 13, 738]. In particular, Kuttner showed that if k is integral, then the single condition

$$\int_0^t x^k|\mu^{(k+1)}(x)|dx=O\{\mu(t)\}$$

was not only sufficient but also necessary for the validity of the proposition: $(R, \lambda, k)\sum a_n=A \Rightarrow (R, \varphi, k)\sum a_n=A$. The author here considers the analogues of the above results for absolute Riesz summability. The analogue of Hardy's theorem was first considered by the reviewer [J. Indian Math. Soc. (N.S.) 6 (1942), 168–180; MR 5, 63] and no analogues have so far existed for Hirst's and Kuttner's results. In the case of integral k, the author here obtains a refinement of the reviewer's result, and, in fact, with a condition on μ which is less stringent than Kuttner's (which shows perhaps that the case of "absolute summability" is not particularly difficult). In the case of non-integral k {the reviewer's proof of the analogue of Hardy's theorem requires modification, as was observed by K. Chandrasekharan and S. Minakshisundaram, in "Typical means", Oxford, 1952, p. 50; MR 14, 1077} the author proves an analogue of Kuttner's theorem under somewhat different conditions. *K. Chandrasekharan.*

Guha, U. Convergence factors for Riesz summability. J. London Math. Soc. 31 (1956), 311–319.

The author proves the following result. Let $\sum a_n$ be summable (R, λ, k), $k\geqq0$, and let $\varphi(x)=\mu\{\lambda(x)\}$, $\varepsilon(x)=\gamma\{\lambda(x)\}$, where (i) $\mu(x)$ is a logarithmico-exponential function, (ii) $1/x\prec\mu'(x)/\mu(x)\prec1$, (iii) $\gamma(x)=\{\mu(x)/x\mu'(x)\}^k$. Then $\sum a_n\varepsilon_n$ is summable (R, φ, k), where $\varepsilon_n=\varepsilon(n)$. He next extends the result to the case in which (ii) is replaced by $1/x\leqslant\mu'(x)/\mu(x)$. With the latter condition, he states the analogue for absolute Riesz summability.

K. Chandrasekharan (Bombay).

Borwein, D. A theorem on Riesz summability. J. London Math. Soc. 31 (1956), 319–324.

The author gives sufficient conditions to ensure the truth of the proposition that $\sum_1^\infty a_n\psi(\lambda_n)$ is summable $(R, \varphi(\lambda_n), k)$ whenever (λ_n) is an unboundedly increasing sequence of positive numbers and $\sum_1^\infty a_n$ is summable (R, λ_n, k). *K. Chandrasekharan* (Bombay).

Ossicini, Alessandro. Sulla sommabilità di Cesaro delle serie di Legendre. Boll. Un. Mat. Ital. (3) 10 (1955), 521–526.

The author proves a theorem on Cesàro summability for Legendre series using an analogous theorem for Fourier series given by S. Yano [Pacific J. Math. 2 (1952), 419–429; MR 14, 267]. In his proof the author makes use of procedures and results contained in Vitali and Sansone, Moderna teoria delle funzioni di variabile reale, parte II [3rd ed., Zanichelli, Bologna, 1952; MR 13, 741] and of a theorem of A. Zygmund [Bull. Internat. Acad. Polon. Sci. Lett. Cl. Sci. Math. Nat. Sér. A. 1927, 309–331, p. 326]. Misprint noted: read $a_n=(2n-1)!!/(2n)!!$ for equation $a_n=(2n+1)!!/(2n)!!$ on page 524. *M. J. De Schwarz.*

Jakimovski, Amnon. Some Tauberian properties of Hölder transformations. Addendum. Proc. Amer. Math. Soc. 8 (1957), 487–488.

Zum Beweis von Satz 3.1 der in der Überschrift genannten Arbeit des Verf. [dieselben Proc. 7 (1956), 354–363; MR 18, 31] wurde ein Satz von Pitt über die reziproke Funktion einer Mellin-Transformation herangezogen, dessen Voraussetzungen unvollständig zitiert waren. Dies wird jetzt verbessert und dann gezeigt, daß Satz 3.1 unverändert richtig bleibt. *D. Gaier* (Stuttgart).

Erdös, P. On a high-indices theorem in Borel summability. Acta Math. Acad. Sci. Hungar. 7 (1956), 265–281. (Russian summary)

Throughout this review the series $\sum a_k$ satisfy a gap condition of the form "$a_k=0$ except if $k=n_j$" where $n_{j+1}-n_j>cn_j{}^{\frac{1}{2}}$ for some constant $c>0$. If $\sum a_k$ is Euler summable, the work of Erdös and of Meyer-König shows that $\sum a_k$ converges [cf. Meyer-König and Zeller, Math. Z. 66 (1956), 203–224; MR 18, 733]. It is not known if the corresponding result for Borel summability is true. But the present paper shows that Borel summability of $\sum a_k$ does imply convergence under the additional hypothesis that the series $\sum 1/(n_{j+1}-n_j)$ converges. The significance of this result is that there is no order condition on the a_k as in Pitt's theorem [Proc. London Math. Soc. (2) 44 (1938), 243–288, Theorem 17] and its extension by Meyer-König [Math. Z. 57 (1953), 351–352; MR 14, 865]. *J. Korevaar* (Madison, Wis.).

See also: Rahman, p. 129; Artemiadis, p. 130.

Approximations, Orthogonal Functions

Pełczyński, A. A generalisation of Stone's theorem on approximation. Bull. Acad. Polon. Sci. Cl. III. 5 (1957), 105–107, X. (Russian summary).

The author announces a generalization of the Stone-Weierstrass theorem. The functions are on a compact

Hausdorff space with values in a linear space whose topology is defined by a countable family of pseudonorms, and polynomials are replaced by multilinear operations. Proofs are to appear in Studia Math.

M. E. Shanks (Princeton, N.J.).

Il'in, V. P. On convergence of sequences of functions in certain functional spaces. Uspehi Mat. Nauk (N.S.) 12 (1957), no. 1(73), 192–195. (Russian)

Let $\{u_i(x_1, \cdots, x_n)\}$ $(i=0, 1, \cdots)$ be a sequence of continuous functions defined in an n-dimensional domain D and having continuous derivatives of order l, summable of order $p\geqq 1$. Suppose that

$$\iint_D |u-u_i|^p dx_1 \cdots dx_n = A_i^p,$$

$$\iint_D \Big[\sum_{i_1, \dots, i_l=1}^{n} \left| \frac{\partial^l (u-u_i)}{x_{i_1} \cdots x_{i_l}} \right|^2 \Big]^{p/2} dx_1 \cdots dx_n = B_i^p.$$

Then by a theorem of S. L. Sobolev [Some applications of functional analysis in mathematical physics, Izdat. Leningrad. Gos. Univ., 1950; MR 14, 565], we have, for instance, the result that if $lp>n$ then $|u-u_i|\leqq C_1A_i+C_2B_i$ throughout D, where C_1 and C_2 are constants independent of u and u_i; in particular, if $\lim_{i\to\infty} A_i = \lim_{i\to\infty} B_i = 0$, then the sequence $\{u_i\}$ converges uniformly to u. The author now lists several extensions of this and similar results. For instance, it is stated that if $lp>n$, $p\geqq 1$, $\lim_{i\to\infty} A_i = 0$, $\lim_{i\to\infty} A_i^{1-n/lp} B_i^{n/lp} = 0$, then the sequence $\{u_i\}$ converges uniformly to u. *E. F. Beckenbach.*

van der Corput, J. G. Asymptotics. IVa, IVb. The multiplication-interpolation method. Nederl. Akad. Wetensch. Proc. Ser. A. 59=Indag. Math. 18 (1956), 129–135, 136–142.

These two papers give a summary of "Asymptotic expansions", rep. IV [1955] by the same author. The purpose is to generalise a variant of the method of Szekeres to determine under general conditions the asymptotic behaviour of functions defined by recurrence relations. [Acta Sci. Math. Szeged 12 (1950), Pars B, 187–198; MR 13, 220]. The advantage of this method is that it is no longer necessary to represent the functions under consideration by integrals or sums, but that the recurrence relation itself is already sufficient for this purpose. The problem is to determine the asymptotic behaviour of the coefficients $\gamma(k)$ in the Maclaurin expansion

$$e^{\omega(w-\psi(tw))} = \sum_{k=0}^{\infty} \gamma(k) w^k.$$

The asymptotic behaviour of the coefficients $\gamma(k)$ is rather complicated. The behaviour of $F(k)=\gamma(k-1)/\gamma(k)$ $(k=1, 2, \cdots)$ is less complicated. A simple non-linear recurrence relation in $F(k)$ is derived, which is easier to handle than the original linear recurrence relation in $\gamma(k)$, and is in turn replaced by a more easily handled "related equation". The author intends to show that the first term of the solution of the recurrence relation for $F(k)$ is given by the solution of the related equation. The paper ends with a simple demonstration of the following generalisation of a result of Szekeres: Let $P(w)$ be a polynomial in w of degree m, with $P'(w)=\sum_{r=1}^{m} p_r w^{r-1}$. Let $g(w)$ be analytic at the origin with $g(0)\neq 0$. Put $e^{P(w)}g(tw)=\sum_{k=0}^{\infty}\gamma(k,t)w^k$ ($\gamma(k,t)$ is a polynomial in t of degree $<k$). Finally, let $F(k,t)=\gamma(k-1,t)/\gamma(k,t)$ $(k\geqq 1)$. Then for sufficiently small $|t|$:

$$\log \frac{\gamma(k,t)}{\gamma(k,0)} =$$

$$\sum_{r=1}^{m} p_r \int_0^t \{F(k,u)\cdots F(k+1-r,u) - F(k,0)\cdots F(k+1-r,0)\}\frac{du}{u}$$

$$(F(k,u)=0 \text{ for } k<0).$$

If the asymptotic behaviour of $F(k,u)$ is given, the right-hand side gives the required asymptotic behaviour of $\log \gamma[(k,t)/\gamma(k,0)]$. *S. C. van Veen* (Delft).

Heflinger, Lee O. The asymptotic behaviour of the Hermite polynomials. Nederl. Akad. Wetensch. Proc. Ser. A. 59=Indag. Math. 18 (1956), 255–264.

The author's doctor's thesis [Univ. of California, 1956] is devoted to applications of the multiplication-interpolation method, discovered by Szekeres [Acta Sci. Math. Szeged 12 (1950), Pars B, 187–198; MR 13, 220] and generalized in the paper reviewed above. The present paper treats only one of the three problems considered in this thesis, namely the determination of the asymptotic behavior of the Hermite polynomial $H_k(x)$ for large $|x|$ and large k, provided that $k<\frac{1}{4}|x|^2$. He obtains for $\log H_k(x)$ an asymptotic expansion of the form

$$k(\log x+\frac{1}{1+\zeta}-\tfrac{1}{2}+\log \tfrac{1}{2}(1+\zeta)+\tfrac{1}{2}\log((1+\zeta)/2\zeta)+\sum_{h\geq 2}' k^{1-h}Q_h(1/\zeta),$$

where $\zeta=(1-4k^2/x)^{\frac{1}{2}}$ and $Q_h(u)$ is a polynomial in u. The method yields an excellent approximation even for small values of k and $|x|$; for instance if we use two terms of the expansion, then for $k=1$, $x=4$ the error is less than 0.004, therefore less than 0.003 times $\log H_1(4)$. The author gives a numerical upper bound for the remainder term which is particularly sharp when only the first few terms are used. The results occurring in this paper are identical with those obtained in the author's thesis with the exception of the numerical upper bound for the remainder term; the upper bound in (1.4) of the present paper requires a supplementary term which is asymptotically equal to zero for large k. *J. G. van der Corput.*

Salzer, H. E. Note on the Fourier coefficients for Chebyshev patterns. Proc. Inst. Elec. Engrs. C. 103 (1956), 286–288.

The author considers the expansion

$$T_n(ax+b)=\sum_{m=0}^{n} b_m T_m(x),$$

where $T_m(x)$ $(m=0, 1, \cdots, n)$ are Chebyshev polynomials, and presents the coefficients b_m in a form convenient for numerical computation. *S. Kulik* (Columbia, S.C.).

Haplanov, M. G. On completeness of certain systems of analytic functions. Rostov. Gos. Ped. Inst. Uč. Zap. no. 3 (1955), 53–58. (Russian)

In earlier papers [Dokl. Akad. Nauk SSSR (N.S.) 80 (1951), 177–180; 83 (1952), 35–38; MR 13, 357; 14, 154] the author gave a matrix method for treating basis and completeness problems of analytic functions. Here he applies his method; we quote two of his results. I. If $f(z)=a_0+a_1z+\cdots$ is regular in $|z|<R$ and is not a rational function, then the set $f_k(z)=a_k+a_{k+1}z+\cdots$

($k=0, 1, \cdots$) is complete in $|z|<R$. II. If $f(z)$ is regular in $|z|<R$, $f(0)\neq 0$ and $\sum_{n=0}^{\infty}|\alpha_n|<\infty$, then the functions $\{z^n f(\alpha_n z)\}_0^{\infty}$ constitute a basis in $|z|<r<R/\sup|\alpha_n|$.

A. Dvoretzky (Princeton, N.J.).

Šnol', È. On connection between Müntz' theorem and orthogonal expansions. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 910–912. (Russian)

The main result proved in this note is the following theorem: If $\psi(x, \lambda_k)$ with $\lambda_k<0$ is a solution of the equation

$$-\psi''+q(x)\psi=\lambda_k\psi \quad (q(x)>0)$$

which is quadratically integrable on $(0, \infty)$, and if $\sum_k |\lambda_k|^{-\frac{1}{2}}=\infty$, then the system of functions $\psi(x, \lambda_k)$ is complete in $L_2(0, \infty)$.

This theorem, which is similar to Müntz's theorem for the system of functions x^{μ_k} ($\mu_k>0$), is applied to prove a uniqueness theorem for a problem in quantum mechanics given by J. Wheeler [Bull. Amer. Phys. Soc. **30** (1955), no. 3, 30].

U. W. Hochstrasser (Lawrence, Kans.).

Paszkowski, S. On the accuracy of approximation with nodes. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 745–748.

Let $\mathscr{C}$ denote the class of all continuous functions on the closed interval $I=[a, b]$ with the norm

$$\|\xi\|=\max_{t\epsilon I}|\xi(t)| \quad (\xi\in\mathscr{C}).$$

Let W_n be the class of algebraic polynomials of degree at most n. Then it is usual to consider the error of best approximation of ξ by polynomials of the class W_n; it is given by

$$\varepsilon_n(\xi)=\inf_{\Psi\epsilon W_n}\|\xi-\Psi\|.$$

One can generalize this. Given the "nodes"

$$T\equiv\{t_1, \cdots, t_m\}$$

such that $a\leqq t_1<t_2<\cdots<t_m\leqq b$, let $W_n(\xi; T)$ be the class of all polynomials ω of degree $n>m$ for which $\omega(t_k)=\xi(t_k)$ for $k=1, 2, \cdots, m$. Define

$$\varepsilon_n(\xi; T)=\inf_{\omega\epsilon W_n(\xi;T)}\|\xi-\omega\|.$$

The author proves that if $m\geqq 3$, and $n\geqq 14[p/c]+12$, where

$$p=\min\{6(b-a), \frac{m-1}{\pi}(2d-(m-1)c)\},$$

$$c=\tfrac{1}{2}\min_{1\leqq k\leqq m-1}(t_{k+1}-t_k),$$

$$d=\max_{1\leqq k\leqq n}\max(t_k-a, b-t_k),$$

and $[p/c]$ denotes the integral part of p/c, then

$$\varepsilon_n(\xi; T)<2\varepsilon_n(\xi)$$

for every $\xi\in\mathscr{C}$.

K. Chandrasekharan (Bombay).

Videnskiĭ, V. S. On uniform approximation in the complex plane. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 169–175. (Russian)

A result of Kolmogorov [Uspehi Mat. Nauk (N.S.) 3 (1948), no. 1(23), 216–221; MR **10**, 35] which was generalized and proved by other methods by Remez [Dokl. Akad. Nauk SSSR (N.S.) **77** (1951), 965–968; Ukrain Mat. Ž. **5** (1953), 3–49; MR **13**, 99; **15**, 407] and Ivanov [Mat. Sb. N.S. **22(70)** (1951), 685–706; **30(27)** (1952), 543–558; MR **13**, 119; **14**, 254] is proved yet more simply and some applications are made. A method employed by T. Hall [Thesis, Univ. of Uppsala, 1950; MR **12**, 23] is used in the proof.

A. Dvoretzky.

Bari, N. K. On locally best approximation of periodic functions by trigonometric polynomials. Moskov. Gos. Univ. Uč. Zap. 181. Mat. **8** (1956), 107–138. (Russian)

Theorems connecting the kth modulus of continuity of a periodic function with its degree of approximation by trigonometric polynomials over a period were stated by Lozinskiĭ [Dokl. Akad. Nauk SSSR (N.S.) **83** (1952), 645–647; MR **13**, 838] but proofs have never been published. The author proves similar results involving pth power approximation and intervals interior to a period, by using an inequality which she proved earlier [Izv. Akad. Nauk SSSR. Ser. Mat. **18** (1954), 159–176; MR **15**, 788]. She also proves several theorems on the partial sums of Fourier series and on conjugate functions. Let $f(x)\in L(0,2\pi)$. The following are typical results from the second group. If, for every $(a', b')\subset(a, b)$, $f\in L^2(a', b')$ and

$$\sum n^{-1}[E_n^{(2)}(f, a', b')]^2<\infty,$$

the Fourier series of f converges almost everywhere on (a, b); here $E_n^{(2)}$ is the best L^2 approximation to f by trigonometric polynomials of degree n. If $W(n)$ increases and $\sum_1^n W(k)/(k+1)=O(W(n))$, $\|S_n(x)\|_{p;(a,b)}=O(W(n))$ implies $\|\bar{S}_n(x)\|_{p;(a',b')}=O(W(n))$ for $[a', b']\subset(a, b)$; here S_n are the partial sums of a Fourier series and the norms are in L^p over the indicated intervals. If σ_n are the arithmetic means of the S_n, and $\omega^{(p)}(f, \delta, a, b)$ is the L^p modulus of continuity of f over (a, b), then, with $0<\alpha<1$, $\omega^{(p)}(f, \delta, a, b)=O(\delta^\alpha)$ implies $\|f(x)-\sigma_n(x)\|_{p;(a',b')}=O(n^{-\alpha})$ and the same with $\bar{f}$ and $\bar{\sigma}_n$. Also $\omega^{(p)}(\bar{f}, \delta, a', b')=O(\delta^\alpha)$.

R. P. Boas, Jr. (Evanston, Ill.).

Kuz'mina, A. L. On the asymptotic representation of polynomials orthogonal on the unit circle. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 793–795. (Russian)

Szegö [Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ., v. 23, New York, 1939; MR **1**, 14] has proved that for polynomials $p_n(z)$, orthonormalized on the unit circle, the asymptotic formula

$$p_n(t)=\frac{t^n}{(2\pi)^{\frac{1}{2}}\bar{D}(t^{-1})}+\varepsilon_n \quad (|\varepsilon_n|<C(\ln n)^{-\varepsilon}),$$

$$D(z)=\exp\left[\frac{1}{4\pi}\int_0^{2\pi}\ln p(\theta)\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta\right],$$

holds uniformly in θ on the circle. In the present note this result is improved for the case when the weight function $p(\theta)$ in the orthogonality relation for the $p_n(z)$ is given by

$$p(\theta)=\lim_{z\to t}|f(z)|^2 \quad (t=e^{i\theta}, 0\leqq\theta\leqq 2\pi),$$

where $f(z)$ is a regular function for $|z|>1$, continuous in $|z|\geqq 1$, having no zero in $|z|\geqq 1$, and with a continuous absolute value on $|z|=1$. In this case it is shown that the asymptotic formula

$$p_n(t)=\frac{t^n}{(2\pi)^{\frac{1}{2}}f(t)}+o(1) \quad (t=e^{i\theta}, 0\leqq\theta\leqq 2\pi)$$

holds uniformly in θ on the unit circle.

U. W. Hochstrasser (Lawrence, Kans.).

Džrbašyan, M. M. On asymptotic approximation by entire functions in a half plane. Dokl. Akad. Nauk SSSR (N.S.) 111 (1956), 749–752. (Russian)

Keldyš a démontré le théorème suivant [C. R.(Dokl.) Acad. Sci. URSS(N.S.) 47 (1945), 239–241; MR 7, 150]: Soit $f(z)$ une fonction holomorphe dans le demi-plan $y>0$, continue sur le demi-plan $y\geq 0$ $(z=x+iy)$; à tout α, $0\leq\alpha<1$, et à tout $\varepsilon>0$ correspond une fonction entière $G_\varepsilon(z)$ telle que $\sup_{y\geq 0}\{|f(z)-G_\varepsilon(z)|e^{|z|^\alpha}\}<\varepsilon$. Ce théorème cesse d'être vrai si $\alpha=1$. L'auteur démontre le théorème suivant: Si f satisfait aux conditions précédentes, et si $P(r)$ satisfait sur $[0,\infty)$ aux conditions 1) $P(r)$ continue, $P(r)$ ne décroit pas, $P(r)/r\downarrow 0$ $(r\to\infty)$, 2) $\int_1^\infty [P(r)/r^2]dr<\infty$, — alors à tout $\varepsilon>0$ correspond une fonction entière $G_\varepsilon(z)$ telle que

$$\text{(A)}\qquad \sup_{y\geq 0}\{|f(z)-G_\varepsilon(z)|e^{P(|z|)}\}<\varepsilon.$$

Le lemme fondamental de l'auteur, ainsi, d'ailleurs, que sa démonstration (utilisation des produits de la forme $\prod(1-e^{2i\alpha_n\omega})/2i\alpha_n\omega)$ sont très semblables à ceux utilisés par S. Mandelbrojt [Duke Math. J. 11 (1944), 341–349; voir aussi Ann. Sci. École Norm. Sup. (3) 71 (1954), 301–320; MR 5, 257; 16, 815]. Si 2) n'a pas lieu, la conclusion cesse d'être vraie; $f(z)$ est une fonction entière même si (A) a lieu pour un seul $\varepsilon>0$. *S. Mandelbrojt* (Paris).

See also: Tietze, p. 151.

Trigonometric Series and Integrals

Chak, A. M. On an analogous Fourier series and its conjugate series. Math. Student 24 (1956), 193–202 (1957).

In this paper the author treats series

$$\tfrac{1}{2}a_0+\sum_{n=1}^\infty a_n(x) \text{ and } \sum_{n=1}^\infty b_n(x),$$

where

$$\frac{a_n}{b_n}(x)=\pm\frac{1}{4}\int_0^\pi [f(x+t)\pm f(x-t)]\,\frac{\cos}{\sin}\left[n\pi\tan\frac{t}{4}\right]dt.$$

Conditions for convergence and summability $(C, 1)$ are shown to be the same as those for Fourier series.

T. Fort (Columbia, S.C.).

Izumi, Shin-ichi; and Satô, Masako. Fourier series. X. Rogosinski's lemma. Kōdai Math. Sem. Rep. 8 (1956), 164–180.

Let $f(x)$ be an integrable function of period 2π and let $s_n(x)$ denote the nth partial sum of its Fourier series. The reviewer proved in 1926 [Schr. Königsberg. Gelehrten Ges. Nat. Kl. 3 (1926), 57–98] that if f is continuous at ξ, then

$$(*)\qquad \frac{1}{2}\{s_n(x_n)+s_n(x_n+\frac{\pi}{n})\}\to f(\xi)$$

whenever $x_n\to\xi$. The authors establish a kind of converse of this result: if f is bounded and if

$$\int_0^t [f(x+u)-f(x-u)]du=o(t)$$

uniformly in some neighborhood of ξ, then (*) implies that f is essentially continuous at ξ. Their proof is based on an analysis of the 'kernel' of the integral expressing the left-hand side of (*). From this they also derive estimates for the Gibbs phenomenon at points of discontinuity which need not necessarily be points of jump. The reviewer found the details of the proofs difficult to follow.

W. W. Rogosinski (Boulder, Colo.).

Kinukawa, Masakiti. On certain strong summability of a Fourier power series. Kōdai Math. Sem. Rep. 9 (1957), 12–22.

The function $f(z)=\sum_0^\infty c_nz^n$ is regular for $|z|<1$ and has a boundary function $f(e^{i\theta})$; $\sigma_n^\delta(\theta)$ is the nth Cesàro sum of order δ of the series $\sum_0^\infty c_ne^{in\theta}$. The author proves that the series $\sum_{n=1}^\infty|\sigma_n^\delta(\theta)-(fe^{i\theta})|^q$ converges for almost all θ when $1<p\leq 2$, $q=p/(p-1)$ and the integral

$$\int_0^1\Delta_1(r)\left(\frac{1}{2\pi}\int_{-\pi}^\pi|f'(re^{i\phi})|^pd\phi\right)^{1/p}dr$$

is finite, where

$$\Delta_1(r)=(1-r)^{1/p-1}\qquad(\delta>1/p-1)$$

$$\Delta_1(r)=(1-r)^{1/p-1}\left(\log\left(\frac{1}{1-r}\right)\right)^{1/p}\qquad(\delta=1/p-1).$$

A second theorem of a similar type is also proved.

H. G. Eggleston (Cambridge, England).

Bhatt, S. N. On negative order summability of a Fourier series. Proc. Nat. Inst. Sci. India. Part A. 22 (1956), 298–304 (1957).

The author extends the convergence criterion for Fourier series of L-integrable functions due to Sunouchi [Tôhoku Math. J. (2) 3 (1951), 216–219; MR 13, 739] to give a criterion for negative-order Cesàro summability. He proves that if $\Delta>1$ and

$$\int_0^t\phi(u)du=o(t^{\Delta+1-1/\Delta})\quad(t\to 0+)$$

$$\int_0^t|d\{u^{\Delta-1+1/\Delta}\phi(u)\}|=O(t)\quad(0\leq t\leq\pi),$$

then the Fourier series of the even function $\phi(u)$ is summable $(C,-1/\Delta)$ at $u=0$. *H. G. Eggleston.*

See also: Ossicini, p. 135; Bari, p. 137.

Integral Transforms

★ **Parodi, Maurice. Introduction à l'étude de l'analyse symbolique. Traité de physique théorique et de physique mathématique, VIII.** Gauthier-Villars, Paris, 1957. viii+246 pp. 3.500 fr.

Cet exposé élémentaire de la transformation de Laplace et de ses applications est standard par son contenu mais très complet, particulièrement clair et d'une présentation parfaite. Il est à la portée d'un lecteur qui serait seulement en possession des éléments de l'analyse classique et qui désire étudier le calcul opérationnel en vue de ses applications. Il fait en quelque sorte le pendant en langue française du "Modern operational mathematics in engineering" [McGraw-Hill, New York, 1944; MR 5, 267] de R. V. Churchill. L'ouvrage contient de nombreux exemples; cependant, on n'y trouve pas d'exercises proposés. L'édition est excellente.

Table des matières: I. Les principes de l'analyse symbolique; II. Application de l'analyse symbolique à la résolution des équations différentielles; III. Deux applications de l'analyse symbolique à la physique; IV. Equations aux dérivées partielles et analyse symbolique;

V. Application de l'analyse symbolique au calcul d'intégrales définies et à l'étude de certaines fonctions; VI. Équations intégrales et analyse symbolique; VII. Étude des réseaux électriques: méthode de Carson et transformation de Laplace. *H. G. Garnir* (Liège).

★ **Doetsch, Gustav. Anleitung zum praktischen Gebrauch der Laplace-Transformation. Mit einem Tabellenanhang von Rudolf Herschel.** R. Oldenbourg, München, 1956. 198 pp. DM 22.00.

This book is addressed to electrical engineers and others who want to apply Laplace transforms to the solution of practical problems. Chapters 1 and 2 describe the rules for manipulating transforms, Chapters 3 to 6 the rules for using transforms to solve ordinary and partial differential equations, difference equations, and integral equations. Chapter 7 describes how to find $F(t)$, by contour integration or other methods, when its transform

$$f(s)=\int_0^\infty F(t)e^{-st}dt$$

is known, and Chapter 8 how to infer the asymptotic behaviour of $F(t)$ as $t\to\infty$ from a knowledge of $f(s)$. An appendix lists pairs $(f(s), F(t))$, mainly with $F(t)$ an exponential polynomial, step function or piecewise linear function, and it also contains a table of explicit solutions for linear differential equations of order ≤ 3 with constant coefficients and simple right-hand sides. Throughout the book, the underlying theory is sketched very lightly (usually without proofs), but the various techniques are carefully explained and are illustrated by worked examples. *G. E. H. Reuter* (Manchester).

Delange, Hubert. Sur les points singuliers de la fonction définie par une intégrale de Laplace-Stieltjes. J. Analyse Math. **5** (1956/57), 1–33.

Let $\alpha(t)$ be a complex-valued function defined on the interval $0\leq t<\infty$ and of bounded variation on every finite subinterval, and let the Laplace-Stieltjes transform

$$f(s)=\int_0^\infty e^{-st}d\alpha(t)$$

converge for Re $s>0$. Let $h_n(y)$ $(n=1, 2, \cdots)$ be complex-valued functions, defined and of bounded variation on the closed real interval $[-l, l]$, such that

$$\limsup_{n\to\infty}\left[\int_{-l}^{l}|dh_n(y)|\right]^{1/n}\leq 1.$$

Set

$$H_n(t)=\int_{-l}^{l}e^{-iyt}dh_n(y).$$

Theorem: If in the above situation $f(s)$ is analytic on the segment $[-il, il]$, then for all numbers x, λ, μ for which $x>0$ and $0<\lambda<1<\mu$, one has

$$\limsup_{n\to\infty}\left|\frac{1}{n!}\int_{\lambda n/x}^{\mu n/x}e^{-xt}t^nH_n(t)d\alpha(t)\right|^{1/n}<\frac{1}{x}.$$

This theorem is used to establish, in slightly generalized form, results on singularities of Laplace-Stieltjes transforms [Bull. Sci. Math. (2) **77** (1953), 141–168; MR **15**, 620] previously obtained by the author by quite different methods. *T. A. Botts* (Charlottesville, Va.).

Gumowski, Igor. Relation entre un critère de réalisabilité, et les transformations fonctionnelles. C. R. Acad. Sci. Paris **244** (1957), 1466–1468.

Lakshmana Rao, S. K.; and Bhatnagar, P. L. A note on the Gegenbauer transform. J. Indian Inst. Sci. Sect. A. 38 (1956), 249–255.

Si considerano due diverse trasformazioni di Gegenbauer, associando ad una $f(x)$ definita in $(-1, 1)$ la successione di numeri

$$(1)\qquad f_1^\nu(n)=T_1\{f(x)\}=\int_{-1}^{1}f(x)(1-x^2)^{\nu-\frac12}C_n^\nu(x)dx$$

oppure

$$(2)\qquad f_2^\nu(n)=T_2\{f(x)\}=\int_{-1}^{1}f(x)C_n^\nu(x)dx,$$

ove $C_n^\nu(x)$ è il polinomio di Gegenbauer di ordine n. Si dimostrano, sotto opportune ipotesi per la $f(x)$, alcune formule che forniscono le espressioni di

$$T_1\left\{(1-x^2)^{-(\nu-\frac12)}\frac{d}{dx}\left[(1-x^2)^{\nu+\frac12}\frac{df}{dx}\right]\right\},$$

$$T_2\left\{\frac{d}{dx}\left[(1-x^2)\frac{df}{dx}+(2\nu-1)xf(x)\right]\right\},$$

$$T_1^{-1}\{-f_1^\nu(n)/[n(n+2\nu)]\},\quad T_2\left\{\frac{df}{dx}\right\},\ T_2\left\{\int_{-1}^{x}f(t)dt\right\}.$$

Segue un'applicazione al calcolo delle frequence proprie nelle vibrazioni di una corda ruotante uniformemente attorno ad un suo estremo, nell'ipotesi che la densità sia variabile ed espressa da

$$\rho(x)=\rho_0(l^2-x^2)^{\nu-\frac12}\qquad(\nu\geq\tfrac12,\ 0\leq x\leq l).$$

A. Ghizzetti (Roma).

★ **Zygmund, A. Hilbert transforms in E^n.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 140–151. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

In this expository lecture the author gives an account of the work on n-dimensional Hilbert transforms contained in the following papers by A. P. Calderón and himself: Acta Math. **88** (1952), 85–139; Trans. Amer. Math. Soc. **78** (1955), 209–224; Studia Math. **14** (1954), 249–271 [MR **14**, 637; **16**, 816, 1017]. A number of additional results in the same field are described. *F. Smithies.*

See also: Pistoia, p. 127; Delavault, p. 210.

Ordinary Differential Equations

★ **Bieberbach, Ludwig. Einführung in die Theorie der Differentialgleichungen im reellen Gebiet.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956. viii+281 pp. DM 29.80.

L'opera, a carattere istituzionale, costituisce una assai efficace introduzione alla teoria delle equazioni differenziali ordinarie nel campo reale, anche perchè in essa trovano posto risultati e indirizzi fra i più recenti. L'esposizione chiara e rigorosa e il notevole numero di esempi, accuratamente elaborati, fanno di questo volume una trattazione eccellente fra le moltissime dedicate allo stesso argomento.

L'introduzione e il primo capitolo sono dedicati all'esistenza e all'unicità delle soluzioni per le equazioni e i sistemi differenziali di forma normale. Dopo l'esposizione del classico metodo delle approssimazioni successive di

Peano-Picard per le equazioni "lipschitziane", vengono considerate equazioni con secondo membro soltanto continuo. Per queste il teorema di esistenza è ottenuto mediante il procedimento dei poligoni approssimanti e l'impiego del teorema di compattezza di Ascoli-Arzelà. L'estensione ai sistemi è immediatamente ottenuta con l'impiego dell'algebra dei vettori. Vengono quindi considerati i teoremi d'unicità generalizzati e la dipendenza delle soluzioni dalle condizioni iniziali e da parametri.

Il capitolo secondo, dopo alcuni brevi cenni ai metodi di integrazione numerica, è dedicato ai procedimenti classici d'integrazione per le equazioni di tipo particolare.

Nel terzo capitolo vengono trattati i sistemi cosidetti "stazionari" (secondo altri autori "autonomi" oppure "conservativi"). Dopo uno studio generale delle caratteristiche e dei punti critici, viene esposta la nota classificazione di tali punti nel caso lineare e i risultati sono immediatamente estesi al caso nel quale le parti lineari siano dominanti nell'intorno del punto critico. Segue un approfondito studio sulla esistenza delle soluzioni cicliche e sui relativi problemi di stabilità. Nell'ultima parte del capitolo sono considerati i sistemi "quasi stazionari".

Il quarto capitolo è dedicato ai problemi al contorno per le equazioni differenziali lineari del secondo ordine. Particolarmente interessante l'accurata esposizione del comportamento asintotico delle autosoluzioni.

L'ultimo capitolo tratta le equazioni alle derivate parziali lineari del primo ordine e vengono esposti i classici metodi d'integrazione a queste relativi, fondati sulla risoluzione di sistemi differenziali ordinari. *G. Fichera.*

Wintner, Aurel. On non-constant Lipschitz factors in the uniqueness problem of ordinary differential equations. Arch. Math. **7** (1957), 465–468.

Consider the initial value problem $x'=f(t,x)$, $x(0)=0$, where x and f are vectors with n real components, and $f(t,x)$ is continuous on the product space of an interval $0\leqq t<a$ and a sphere $|x|<b$. A function $\lambda(t)$, defined on an interval $0<t\leqq t_0$ $(<a)$, is called a Lipschitz function if the condition

$$|f(t,x_1)-f(t,x_2)|\leqq\lambda(t)|x_1-x_2| \quad (|x_1|,\,|x_2|<b;\,0<t\leqq t_0)$$

implies the uniqueness of the solution of the problem on an interval $0\leqq t<c$, which depends in general upon the choice of $f(t,x)$. This paper discusses properties of Lipschitz functions, the principal result being the following theorem: If $\lambda(t)$ $(0<t\leqq t_0)$ is a non-negative continuous function satisfying the condition

$$\limsup_{t\to 0}\,[1+\lambda(t)]e^{-\mu(t)}>0,$$

where $\mu(t)=\int_t^{t_0}\lambda(s)ds$ $(t\neq 0)$, then $\lambda(t)$ is a Lipschitz function. The following further result is obtained as a corollary: In order that a non-negative continuous function $\lambda(t)$ $(0<t\leqq t_0)$ be a Lipschitz function, it is sufficient that the following condition be violated:

$$\log t+\int_t^{t_0}\lambda(s)ds\to\infty,\text{ as } t\to 0.$$

L. A. MacColl (New York, N.Y.).

Čečik, V. A. Approximate method of solution of singular differential equations. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 517–520. (Russian)

The problem is to find a solution of $y_k'=f_k(x,y_1,\cdots,y_n)$ $(k=1,\cdots,n)$ which satisfies $\lim_{x\to 0+}y_k(x)=0$. The f_k are assumed to be continuously differentiable, although not necessarily bounded, in $0<x\leqq b$, $|y_k|\leqq a$; furthermore, a continuous summable function $\varphi(x)$ in $(0,b]$ is assumed to exist such that $|f_k(x,y_1,\cdots,y_{k-1},0,y_{k+1},\cdots,y_n)|$, $f_{k,y_k}(x,y_1,\cdots,y_n)$ and $|f_{k,y_i}(x,y_1,\cdots,y_n)|$, $i\neq k$, are all $\leqq\varphi(x)$. In this case, Picard's method cannot be applied in general. A method of successive approximations based on the ideas of Čaplygin is devised, the solution appearing as the limit of a pair of monotonic sequence of approximating functions. With supplementary assumptions it is possible to show that the rate of convergence is at least geometric or exponential. The details are very difficult to summarize. *J. L. Massera* (Montevideo).

Krasnosel'skiĭ, M. A.; and Kreĭn, S. G. On the theory of ordinary differential equations in Banach spaces. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. no. 2 (1956), 3–23. (Russian)

Slight extensions and detailed proofs of results stated by the same authors in Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 13–16 [MR **17**, 151]. *F. A. Ficken.*

Slugin, S. N. An unrestrictedly applicable method of Čaplygin type for ordinary differential equations of nth order. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 936–939. (Russian)

For the mth order equation

(1) $y^{(m)}-f(t,y,y',\cdots,y^{(m-1)})\equiv y^{(m)}-f[y]=0$, $y^{(k)}(t_0)=\alpha_k$,

the author assumes that $B\leqq\partial f/\partial y^{(k)}\leqq A_k$ $(B\leqq 0\leqq A_k;$ $k=0,\cdots,m-1)$ and defines

$$\Gamma(\Delta y^{(m)})=\Delta y^{(m)}-\sum_{k=0}^{m-1}A_k\Delta y^{(k)},\quad \Delta y^{(k)}(t_0)=0,$$

$$P(y^{(m)})=y^{(m)}-f[y].$$

If $\underline{y}_0^{(m)}(t)\leqq\bar{y}_0^{(m)}(t)$, $\underline{y}_0^{(k)}(t_0)=\bar{y}_0^{(k)}(t_0)=\alpha_k$, and

$$(\bar{y}_0-\underline{y}_0)^{(m)}-\sum_{k=0}^{m-1}A_k(\bar{y}_0-\underline{y}_0)^{(k)}\geqq\left\{\begin{array}{l}\bar{y}_0^{(m)}-f[\bar{y}_0],\\ -\underline{y}_0^{(m)}+f[\underline{y}_0],\end{array}\right.$$

then the iterative procedure $y_{n+1}^{(m)}=y_n^{(m)}-\Gamma^{-1}P(y_n^{(m)})$ applied to $\underline{y}_0$ and $\bar{y}_0$ generates sequences which converge monotonically to the solution of (1). This result is a special case of an iterative theorem in functional analysis.

J. P. LaSalle (Notre Dame, Ind.).

Heading, J. The Stokes phenomenon and certain nth-order differential equations. I. Preliminary investigation of the equations. Proc. Cambridge Philos. Soc. **53** (1957), 399–418.

Heading, J. The Stokes phenomenon and certain nth order differential equations. II. The Stokes phenomenon. Proc. Cambridge Philos. Soc. **53** (1957), 419–441.

In the first paper the author obtains solutions of the differential equation $d^nu/dz^n=(-1)^nz^mu$, $m+n\neq 0$, in form of power series, integrals of the Mellin-Barnes type, and integrals of Laplace's type. He uses the integral representations to obtain the leading terms of the asymptotic expansions of certain solutions for $z\to\infty$ in certain sectors of the complex plane. He also indicates the corresponding results for the differential equation

$$d^nu/dz^n=(-1)^n(e^{az}-1)u.$$

The object of the second paper is to consider the asymptotic expressions in greater detail and to extend them to all values of arg z. In this process, the Stokes phenomenon is encountered. The author first discusses the

Stokes phenomenon in connection with the Airy integral in a manner which generalizes to his nth order equations. He then defines Stokes lines and anti-Stokes lines. The former are lines along which one of the exponential functions occurring in the asymptotic representations has a negative real exponent, and the latter occur half-way between the former. These lines are represented in a diagram in which the dominancy or subdominancy of the various asymptotic solutions is also noted. This diagram is used in the determination of the Stokes multipliers for the asymptotic solutions of the differential equation

$$(\theta-p_1)(\theta-p_2)\cdots(\theta-p_n)u=e^{n\pi i}wu,\ \theta=w(d/dw).$$

A. Erdélyi (Pasadena, Calif.).

Sibuya, Yasutaka. Sur un système d'équations différentielles ordinaires à coefficients presque-périodiques et contenant des paramètres. J. Fac. Sci. Univ. Tokyo. Sect. I. **7** (1957), 407–417.

The author considers a system of differential equations of the first order, of which the right-hand sides are nonlinear expressions in the dependent variables, containing also the independent variable. These expressions are continuous and of a common period as regards the independent variable. This paper shows how a set of bounded solutions of the system of differential equations may be constructed. In a previous paper [same J. **7** (1956), 333–341; MR **18**, 38] it was shown that the solutions of the system of differential equations approach almost periodic functions as $t\to+\infty$ under appropriate assumptions. So, a set of solutions must be found which approach such almost periodic functions as $t\to+\infty$. In order to achieve this, a corresponding linear system of differential equations of the first order is considered, first of homogeneous and then of non-homogeneous type. The original non-linear differential equations are then reduced by a formal almost-periodic transformation to a form admitting of an iterative solution, starting from the linear system. The convergence of the iterative solution is shown. This has the desired properties. *M. J. O. Strutt* (Zurich).

Richard, Ubaldo. Alcuni problemi asintotici per le equazioni differenziali lineari. Univ. e Politec. Torino. Rend. Sem. Mat. **15** (1955–56), 59–64.

A brief survey of a graduate course on the subject of the title, with indications on certain unsolved questions which may serve as subjects of doctoral theses. *J. L. Massera.*

Utz, W. R. A note on second-order nonlinear differential equations. Proc. Amer. Math. Soc. **7** (1956), 1047–1048.

The following results are proved: 1. If $\alpha(x)\leqq 0$, $\beta(x)>0$ are real functions, a solution $x(t)$ of $x''+\alpha(x)x'+\beta(x)x=0$ which exists when $t\to\infty$ either oscillates (vanishes infinitely often) or becomes monotone; in the latter case, its limit as $t\to\infty$ is positive or negative, respectively, when x is increasing or decreasing. 2. Let c be a positive constant and $g(z)\leqq 0$ a real function, $g(0)=0$; a solution $x(t)$ of $x''+g(x')+cx=0$ either oscillates or approaches $+\infty$ or approaches $-\infty$ when $t\to\infty$. *J. L. Massera.*

Feldmann, László. On a characterization of the systems of classical orthogonal polynomials. Magyar Tud. Akad. Mat. Fiz. Tud. Oszt. Közl. **6** (1956), 87–92. (Hungarian)

The author proves a conjecture of the reviewer: The classical (Jacobi, Laguerre, Hermite) polynomial systems are the only orthogonal ones which satisfy differential equations (1) $a_n(x)y_n''+b_n(x)y_n'+c_n(x)y_n=\lambda_n y_n$ of the Sturm-Liouville type. The proof is rather tricky and makes exclusive use of the orthogonal polynomial systems' property that the zeros of y_n are separated by those of y_{n+1}. It is proved that (1) can be reduced to

$$(2)\qquad (ax^2+bx+c)y''_n+xy_n'=\lambda_n y_n,$$

and that it has polynomial solutions of the type described above only if (*) $a\geqq 0$, $c<0$, $\lambda_n>0$. To finish the proof it is necessary to show that all (2) with (*) have only classical polynomials as solutions. Instead, the author merely proves that the classical polynomials satisfy (2) with (*), notwithstanding that the proof can be completed in the manner mentioned above by a result of the author in Publ. Math. Debrecen 3 (1954), 297–304 [MR **17**, 361]. *J. Aczél* (Debrecen).

Feldmann, L. On a characterization of the classical orthogonal polynomials. Acta Sci. Math. Szeged **17** (1956), 129–133.

This is an English version of the paper reviewed above. Here the deficiency of the proof mentioned there is eliminated. There are also other minor deviations in the text. *J. Aczél* (Debrecen).

Moore, Richard A. The least eigenvalue of Hill's equation. J. Analyse Math. **5** (1956/57), 183–196.

The functional dependence of the least eigenvalue $(-a)$ of Hill's equation

$$y''+(-a+bq(x))y=0$$

on the parameter b is investigated. Here q is a periodic function of period one with mean zero, and y is required to be periodic with period one.

By using various Sturmian theorems, the author proves the following and other similar properties of the function $a(b)$: $a(b)$ is convex, $a(0)=a'(0)=0$, and

$$a(b)\leqq\begin{cases} mb-m^2w^{-2} & (b\leqq 2mw^{-2})\\ \frac{1}{4}w^2b^2 & (2mw^{-2}\leqq b\leqq 2Mw^{-2})\\ Mb-m^2w^{-2} & (b\geqq 2Mw^{-2})\end{cases}$$

where m and M are the minimum and maximum values of q, while w is the maximum value of $|\int_{\xi_1}^{\xi_2}q(x)dx|$ for $0\leqq\xi_1\leqq\xi_2\leqq 1$. *H. F. Weinberger.*

Mikusiński, J. Sur l'équation $x^{(n)}+A(t)x=0$. Ann. Polon. Math. **1** (1955), 207–221.

The author discusses the differential equation in the title, and provides theorems on the nature of certain solutions. There are difficulties with Théorème 11 and its corollary. In the former, the author appears to assume that the limit of a sequence of continuous functions which are positive on $a<x<b$ and zero at a and at b necessarily has the same properties. This objection appears to be non-trivial when $n=4$, for example. The conclusion of Corollaire 3 is incorrect, as the example $x^{(4)}+4x=0$ shows. Indeed, the solution defined by $x(0)=x'(0)=x''(0)=0$, $x'''(0)=1$ may be readily seen to have its first positive zero at the first positive zero of the equation $\tanh t=\tan t$. This zero is approximately 3.926, whereas the corollary states it must be $\leqq 2\sqrt{\pi}$. *W. Leighton.*

Westfold, K. C. The solution of linear vector differential equations containing gyroscopic terms. Amer. Math. Monthly **64** (1957), 174–180.

A solution is given of vector differential equations of

the type governing the motion of Foucault's pendulum; the method, though related to Milne's "Vectorial mechanics" [Interscience, New York, 1948; MR 10, 488], is more systematical. Use is made of the theory of characteristic vectors; there are three distinct eigenvalues, two of which are complex. *O. Bottema* (Delft).

André, Johannes. Eine Bemerkung über unstetige Regelungen mit Stellungszuordnung. Z. Angew. Math. Mech. 36 (1956), 268–269.

Necessary and sufficient conditions are found that an arbitrary solution of $dz/dt=Dz+R\operatorname{sgn}(Sz)$ [cf. the paper reviewed below] should tend to the origin more rapidly than if the term $R\operatorname{sgn}(Sz)$ were omitted. It is assumed that z is a 2-vector, R is a constant $2\times k$ matrix, S is a constant $k\times 2$ matrix, and that $zDz<0$ for all $z\neq 0$. *D. C. Lewis, Jr.* (Baltimore, Md.).

Seibert, Peter. Über unstetige Regelungen dynamischer Systeme mit mehreren Freiheitsgraden. Z. Angew. Math. Mech. 36 (1956), 288–289.

Theories of discontinuous automatic control systems lead to equations of the form $dz/dt=Dz+R\operatorname{sgn}(Sz)$, where z is an n-vector, D is a constant $n\times n$ matrix, R is a constant $n\times k$ matrix, and S is a constant $k\times n$ matrix. The right-hand member is discontinuous when one or more components of Sz are zero. Attention in this paper is confined to the case $k=1$, so that the locus of discontinuities is a single hyperplane H. On account of the discontinuity on H, the usual existence theorems for differential equations do not apply. At certain points z_0, called "end points", there is no solution, $z(t)$, for $t\geqq 0$, such that $z(0)=z_0$, but only for $t<0$. The equations only approximate the actual phenomenon of practice, since there is always a slight time delay after the system reaches H before the automatic control is actuated. This gives rise to small oscillations about H approximating a curve drawn on H. The author gives necessary and sufficient conditions that this curve should tend to the origin faster than it would do if the term $R\operatorname{sgn}(Sz)$ were absent. *D. C. Lewis, Jr.* (Baltimore, Md.).

Barbašin, E. A. On two schemes for proving theorems of stability in the first approximation. Dokl. Akad. Nauk SSSR (N.S.) 111 (1956), 9–11. (Russian)

Zadiraka, K. V. A new proof of the theorem of Tihonov. Dopovidi Akad. Nauk Ukrain. RSR 1956, 223–226. (Ukrainian. Russian summary)

The author considers the question of the closeness of the solutions of the system of differential equations

$$\frac{d\mathbf{x}}{dt}=\mathbf{f}(\mathbf{x},\mathbf{z},t),\quad \mathbf{x}(t_0)=\mathbf{x}_0,$$

$$\frac{\mu d\mathbf{z}}{dt}=\mathbf{F}(\mathbf{x},\mathbf{z},t),\quad \mathbf{z}(t_0)=\mathbf{z}_0$$

(where μ is a small parameter) to the solutions of the corresponding degenerate system [see Vasil'eva, Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk 9 (1954), no. 3, 29–40; MR 16, 362]. The author's theorem is a special case of previously known theorems. The author refers exclusively to early articles of the reviewer [Dokl. Akad. Nauk SSSR (N.S.) 65 (1949), 789–792; MR 10, 708] and of A. N. Tihonov [Mat. Sbornik N.S. 22(64) (1948), 193–204; MR 9, 588], ignoring all their later work, some of which contains results that are more general than the author's. *I. S. Gradštein* (RŽMat 1957, no. 3119).

Voznyuk, L. L. On the stability of periodic solutions of high order equations. Dopovidi Akad. Nauk Ukrain. RSR 1957, 13–17. (Ukrainian. Russian and English summaries)

The stability of the periodic solution of the equation

$$(1)\qquad R(p)z=\varepsilon\Phi(z,\varepsilon)$$

is considered. Here ε is a positive parameter, $p=d/dt$, $R(p)$ is an analytic function of p, $\Phi(z,\varepsilon)$ has a sufficient number of derivatives with respect to z and is analytic in ε.

The equation of variations is constructed for equation (1) by the Kryloff-Bogolyuboff method. The characteristic equation is then constructed, and the stability of the periodic solution is discussed on the basis of the nature of the roots of the characteristic equation. A theorem is proved regarding the analytical dependence of the solution of the equation of variations upon the value of ε. *H. P. Thielman* (Ames, Ia.).

Gillies, A. W. The periodic solutions of the differential equation of a resistance-capacitance oscillator. Quart. J. Mech. Appl. Math. 10 (1957), 101–121.

The equation considered is normalized to the form

$$\{(D^2+1)\left(D+\frac{\sqrt{6}}{5}\right)+\varepsilon D^3\}x+g(D)\sum_{n=2} c_n\mu^{n-1}x^n=g(D)2B\cos\omega t,$$

where ε and μ are small parameters which in the main part of the discussion are related by $\varepsilon=\mu^2$, and $g(D)$ is a particular polynomial operator of the third degree. The c_n are assumed to be $O(1)$, B and $\omega-1$ are taken as $O(\varepsilon)$. The periodic solutions having the period of the forcing term are obtained by the method which was previously applied to a second-order equation with unsymmetrical non-linear damping [same J. 8 (1955), 107–128; MR 16, 926]. It is shown that the resonance curves are of the same form as those obtained for the second-order equation and that the stability of the periodic solutions is determined by variational equations which are likewise identical in form to those previously obtained for the second-order equation. *M. Zlámal* (Brno).

Cesari, L.; and Hale, J. K. A new sufficient condition for periodic solutions of weakly nonlinear differential systems. Proc. Amer. Math. Soc. 8 (1957), 757–764.

The authors give sufficient conditions for the existence of periodic solutions of

$$\ddot{y}_i+\sigma_i^2y_i=\varepsilon f_i(y_1,\cdots,y_n,\dot{y}_1,\cdots,\dot{y}_n,\varepsilon;t)\quad (i=1,2,\cdots,n)$$

such that $y_i(-t)=y_i(t)$ $(i=1,2,\cdots,m)$, $y_i(-t)=-y_i(t)$ $(i=m+1,\cdots,n)$. The conditions, besides those which ensure periodic solutions, are certain symmetry properties of the f_i $(i=1,2,\cdots,n)$. The methods are those of R. A. Gambill and J. K. Hale [J. Rational Mech. Anal. 5 (1956), 353–394; MR 17, 1086]. *C. E. Langenhop.*

Lykova, O. B. On one-frequency oscillations in systems with many degrees of freedom. Dopovidi Akad. Nauk Ukrain. RSR 1957, 8–12. (Ukrainian. Russian and English summaries)

The author considers the system of differential equations

$$\frac{dx_k}{dt}=X_k(x_1,\cdots,x_n)+\varepsilon X_k(t,x_1,\cdots,x_n,\varepsilon)$$

($k=1, 2, \cdots, n$), where ε is a small parameter.

It is assumed that for the system of unperturbed equations there is known a two parameter family of periodic solutions

$$x_k = x_k^0(\omega t+\varphi, a) \quad (k=1, 2, \cdots, n)$$

of period $2\pi/\omega$ in t. Furthermore, it is assumed that the functions on the right-hand side of the equations of the system are analytic, regular in the neighborhood of the curves

$$x_k = x_k^0(\omega t+\varphi, a);\ \varepsilon=0 \quad (k=1, 2, \cdots, n)$$

in the $(n+1)$-dimensional space, periodic in t with period 2π, and that the characteristic equation has a pair of purely imaginary roots, $\pm\lambda i$, while the other roots have negative real parts. Under these assumptions a two-parameter family of approximate particular solutions is constructed which for small enough ε is near (in the orbital sense) to the family of periodic solutions. *H. P. Thielman.*

Berštein, I.; and Halanaĭ, A. Index of a singular point and the existence of periodic solutions of systems with small parameter. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 923–925. (Russian)

Let (1) $\dot{x}=X(x)+\mu Y(x, t, \mu)$, x, X, Y being n-vectors, Y periodic in t of period ω, $X(0)=0$; it is assumed that conditions ensuring existence, uniqueness and continuous dependence of the solutions on the initial values and μ are satisfied. The following simple but apparently new theorem is proved: If 0 is an isolated singular point of the vector field X and has a non-vanishing index, and if equation (1′) $\dot{x}=X(x)$ has no periodic solutions in a neighborhood of 0, then (1) has periodic solutions of period ω for sufficiently small μ. The last assumption may be partially avoided (but not entirely omitted, as examples show) in the following cases: (a) if in a neighborhood of 0 there are no periodic solutions of (1′) with period $\leqq\omega$; (b) if $n=2$ and (1′) has no periodic solutions of period ω. *J. L. Massera* (Montevideo).

Sansone, Giovanni. L'equazione di T. Uno e R. Yokomi. Rend. Sem. Mat. Fis. Milano **26** (1954–55), 9 (1957).

A brief summary of some results which the author, together with R. Conti, has presented in full in other papers [Ann. Mat. Pura Appl. (4) **37** (1954), 37–59; **38** (1955), 205–212; MR **16**, 478; **17**, 39]. *L. A. MacColl.*

Saito, Tosiya. On the system of non-linear differential equations with periodic coefficients. Kōdai Math. Sem. Rep. **8** (1956), 97–106.

Let $\dot{x}=f(x, t)$, where f is analytic in x in the neighborhood of $x=0$ and periodic in t of period 1. The author proves in detail: if the real parts of the characteristic exponents λ_i of the linear approximation are all positive or all negative, the solutions are given by

$$x(t)=\sum P_{k_1\cdots k_n}(t)\exp\{(k_1\lambda_1+\cdots+k_n\lambda_n)t\},$$

where the P's are polynomials in t with periodic coefficients. This theorem has already been essentially stated and proved by Lyapunov [Problème général de la stabilité du mouvement, Princeton,1 947; MR **9**, 34]. *J. L. Massera* (Montevideo).

Lyaščenko, N. Ya. An analogue of the theorem of Floquet for a special case of linear homogeneous systems of differential equations with quasi-periodic coefficients. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 295–298. (Russian)

The author obtains a representation for the solution of linear systems of differential equations with quasi-periodic coefficients analogous to the known representation for the case of periodic coefficients. The method depends upon a reduction to diagonal form of the coefficient matrix by means of quasi-periodic matrices, permissible under certain conditions, and then the integration of quasi-periodic functions. Under appropriate conditions which prevent the phases from being too strongly dependent, it is shown that these integrals have the appropriate form. *R. Bellman.*

Reissig, Rolf. Periodische Erregung eines einfachen Schwingers mit Selbststeuerung. Math. Nachr. **15** (1956), 181–190.

The equation $\ddot{x}+F(\dot{x})+G(x)=E(t)$, where $E(t)$ is periodic with period T, is proved (under suitable conditions on E, F, G) to admit a periodic solution with period T. The method, based on the Brouwer fixed-point theorem, is obvious in principle, but involves, as is usual with problems of this type, a lot of detail to set up the problem in a form to which the fixed point theorem can be applied. *D. C. Lewis, Jr.* (Baltimore, Md.).

Reissig, Rolf. Selbsterregung eines einfachen Schwingers. Math. Nachr. **15** (1956), 191–196.

It is proved, under suitable conditions, that the equation $\ddot{x}+F(\dot{x})+G(x)=0$ admits a periodic solution. The method of proof involves the well-known theorem of Bendixson in the usual way. The difficulties are similar to those of the paper reviewed above and are handled in a similar way. The reader should be referred to a paper by N. Levinson and O. K. Smith [Duke Math. J. **9** (1942), 382–403; MR **4**, 42] where a slightly different and, in some respects, more general problem is treated by quite similar methods. *D. C. Lewis, Jr.* (Baltimore, Md.).

★ **Marcus, Marvin D. An invariant surface theorem for a non-degenerate system.** Contributions to the theory of nonlinear oscillations, vol. 3, pp. 243–256. Annals of Mathematics Studies, no. 36. Princeton University Press, Princeton, N. J., 1956. $4.00.

An invariant surface M of dimension r for

$$(*)\qquad dx/dt=f(x, t)$$

(where x, f are n-vectors and $f\in C^1$) is a C^r manifold in E^n with the property that if $U(t)$ is a solution of (*) and $U(t_0)\in M$, then $U(t)\in M$ for all t [see the paper reviewed below].

In the holonomic system

$$(**)\qquad \begin{aligned} \frac{d\theta}{dt}&=1+zH(\theta, \varphi, z)+\lambda P(\theta, \varphi, z),\\ \frac{d\varphi}{dt}&=1+zK(\theta, \varphi, z)+\lambda Q(\theta, \varphi, z),\\ \frac{dz}{dt}&=A(\theta, \varphi)z+z^2L(\theta, \varphi, z)+\lambda R(\theta, \varphi, z), \end{aligned}$$

suppose $A, H, K, L, P, Q, R\in C^3$ and periodic in θ and φ of periods ω_1 and ω_2 respectively. For $\lambda=0$, the solutions with initial values on the θ and φ axes generate a torus $\Gamma_0=\Gamma(\theta, \varphi)$ of period ω_1 in θ and ω_2 in φ if the planes $\varphi=\pm n\omega_2$, $\theta=\pm n\omega_1$ ($n=0, 1, 2, \cdots$) are identified.

The Schauder fixed-point theorem is used to show that if certain conditions on H, K, and A are satisfied, there exists $\lambda^*>0$ such that for any λ with $|\lambda|<\lambda^*$ there is a family of invariant tori $\Gamma_\lambda(\theta, \varphi)$ for (**). An extension of this result is obtained for an n-dimensional system in which z is an $(n-2)$-vector. *J. Cronin* (New York, N.Y.).

★ **Diliberto, Stephen P. An application of periodic surfaces (solution of a small divisor problem).** Contributions to the theory of nonlinear oscillations, vol. 3, pp. 257–259. Annals of Mathematics Studies, no. 36. Princeton University Press, Princeton, N. J., 1956. $4.00.

Consider $\ddot{x}+r\dot{x}+\omega_1^2x=\varepsilon f(x,\dot{x})+p(t)$, where f is of class C^2 on $E^1\times E^1$, $p(t)=A_2\sin\omega_2t+A_3\sin\omega_3t$ with $\omega_2\neq\omega_3$, and $r>0$. Let $y(t)$ be the (unique) almost periodic solution when $\varepsilon=0$. The author uses a result of the paper reviewed above to show that there exists, for $|\varepsilon|$ small enough, a solution $x(t)=y(t)+z(t,t,\varepsilon)$ with z continuous in t, ε and $z(\theta_2,\theta_3,\varepsilon)$ of period $2\pi/\omega_i$ in θ_i, $z(\theta_2,\theta_3,0)\equiv0$. Moreover, every nearby solution tends to $x(t)$ exponentially and, as $\varepsilon\to0$, $x(t)\to y(t)$ uniformly in t on $(-\infty,+\infty)$.

H. A. Antosiewicz (Providence, R.I.).

Krasnosel'skiĭ, M. A. On the application of the methods of non-linear functional analysis to certain problems of periodic solutions of non-linear mechanics. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 283–286. (Russian)

The author discusses, with many references but no proofs, some methods which can be applied to some non-linear equations, for example,

$$dx_i/dt=f_i(t,x_1,\cdots,x_n)\qquad(i=1,\cdots,n),$$

or

$$d^2x_i/dt^2+g_i(t,x_1,\cdots,x_n,dx_1/dt,\cdots,dx_n/dt)=0\qquad(i=1,\cdots,n).$$

M. M. Day (Urbana, Ill.).

Vorovič, I. I. On certain cases of existence of periodic solutions. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 165–168. (Russian)

Let l_2 be the Hilbert space of real square-summable sequences $X=\{x_i\}$; denote by S_R the solid sphere of radius R, $S_\infty=l_2$. A functional $\phi(X,v,w)$ – $X\in l_2$; v, w real; $-1\leq v,w\leq1$ (interval I) – is said to be (a) continuous if it is weakly continuous, (b) continuously differentiable in $S_R\times I\times I$ if the Fréchet differential $\mathrm{grad}_{l_2}\phi$ exists and has continuous components, and if

$$|\phi(X+H,\sin t,\cos t)-\phi(X,\sin t,\cos t)|\leq \|H\cdot\mathrm{grad}\,\phi\|_{l_2}+\omega(X,H,t),$$

where $|\omega|\leq k(t)\|H\|^2$, k summable in $(0,2\pi)$. Theorems: 1. Assume ϕ is defined in $S_\infty\times I\times I$ and (i) is continuous and continuously differentiable in each $S_R\times I\times I$, (ii) ϕ is even in each argument, (iii) $X\cdot\mathrm{grad}\,\phi\leq0$ with equality sign only if $X=0$; then, for each given $\rho>0$, there are infinitely many values of λ accumulating at $\lambda=0$ such that the equation $\lambda^2\ddot{X}=\mathrm{grad}(X,\sin t,\cos t)$ has odd periodic solutions satisfying $\int_0^{2\pi}\sum\dot{x}_i^2dt=\rho^2\,\{<\rho^2?\}$. 2. Let $\phi=-\frac12\sum\mu_ix_i^2+U(X,\sin t,\cos t)$, $\mu_i\leq\delta<1$, where $U\geq0$ satisfies conditions (i) and (ii) above; let $F(t)$, periodic of period 2π, have components $f_i\in L_2(0,2\pi)$, $\sum\|f_i\|_{L_2}^2<\infty$, $\int_0^{2\pi}f_i\cos mt\,dt=0$; then $\ddot{X}=\mathrm{grad}\,\phi+F(t)$ has at least one odd periodic solution satisfying

$$\int_0^{2\pi}\sum\dot{x}_i^2dt<4(1-\delta^2)^{-1}\int_0^{2\pi}\|F\|^2dt.$$

3. Let $\phi=\frac12\sum\mu_ix_i^2+U(X,\sin t,\cos t)$, $\mu_i\geq\delta>0$, $U\geq0$ satisfying condition (i) above; let $F(t)$, periodic of period 2π, have components $f_i\in L_2(0,2\pi)$, $\sum\|f_i\|_{L_2}^2<\infty$; then $X=\mathrm{grad}\,\phi+F$ has at least one periodic solution. No proofs are given, although they are apparently based on "direct" methods (Bubnov-Galerkin and Ritz).

J. L. Massera (Montevideo).

★ **Kononenko, V. O. On nonlinear oscillations in systems with varying parameters.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 6 pp.

Translation of Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 229–232 [MR **17**, 616].

★ **Kononenko, V. O. On oscillations in nonlinear systems with many degrees of freedom.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 6 pp.

Translation of Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 664–667 [MR **17**, 736].

Chandrasekhar, S.; and Reid, W. H. On the expansion of functions which satisfy four boundary conditions. Proc. Nat. Acad. Sci. U. S. A. **43** (1957), 521–527.

The authors obtain complete orthogonal sets of eigenfunctions for the following two problems. (1) The equation $d^4y/dx^4=\lambda^4y$ with the boundary conditions $y=dy/dx=0$ at $x=\pm\frac12$. (2) The equation

$$\left(\frac{d^2}{dr^2}+\frac1r\frac{d}{dr}-\frac{\nu^2}{r^2}\right)^2y=\alpha^4y$$

with the boundary conditions $y=dy/dr=0$ at $r=1$ and $r=\eta$, where $\eta<1$. Applications of the resulting expansions to various problems of hydrodynamic stability are discussed, and some numerical data are included.

W. Rudin (Rochester, N.Y.).

Sims, Allen R. Secondary conditions for linear differential operators of the second order. J. Math. Mech. **6** (1957), 247–285.

Consider the differential equation $Ky-\lambda y=f(x)$, where $Ky\equiv-y''+qy$, on the interval $I=(a,b)$ or $[a,b]$. The Sturm-Liouville theory deals with the case where q is real and continuous on the closed interval; the Weyl limit-point, limit-circle theory deals with the case where q is real and continuous on the open interval but where a or b may be a singular point. The present paper extends the above theory in two ways, namely (i) by allowing $q(x)$ to be complex-valued, and (ii) by generalizing the usual boundary value problem to problems depending upon "secondary conditions". A secondary condition is defined as follows: Let D^* denote the largest linear manifold for which Ky is defined and in $L^2[a,b]$. If P denotes a property that elements of D^* can have and if D_P is the set of elements of D^* with this property, then P is said to determine a secondary condition for the differential equation if for each f in $L^2[a,b]$ ($f\not\equiv0$) and for at least one λ there exists one and only one y in D_P for which $L_Py-\lambda y=f$, where L_P is the contraction of K having domain D_P. Restrictions requiring that P be such that D_P be a linear set, and that the inverse of $L_P-\lambda I$ be a bounded operator are imposed; the problem is then to find all contractions of K with non-empty resolvent sets. It is assumed, for the most part, that $\mathrm{Im}(q)>0$ $[<0]$ and $\mathrm{Im}(\lambda)<0$ $[>0]$. There are three possibilities for either end-point of (a,b): (I) limit-point but only one solution of class L^2, (II) limit-point and two solutions of class L^2 (unlike the situation for the Weyl theory), and (III) limit-circle and two solutions of class L^2. Thus, essentially six

situations for the interval (a, b) can occur; case (I) at a and cases I, II or III at b, case II at a and cases II or III at b, and finally case III at both a and b. It turns out, as in the earlier theory, that the particular one of the above situations is determined by $q(x)$ only (and not by λ).

C. R. Putnam (Lafayette, Ind.).

Norkin, S. B. Boundary problem for a second order differential equation with a retarded argument. Moskov. Gos. Univ. Uč. Zap. **181.** Mat. 8 (1956), 59–72. (Russian)

The author establishes the existence and asymptotic behavior of characteristic values associated with the equation

$$y''(x)+\lambda y(x)+M(x)y(x-\Delta(x))=0$$

and the boundary conditions $y(0)=0$, $(y'(0)<\infty)$, $y(\pi)=0$, $y(x-\Delta(x))=0$, $x-\Delta(x)\leqq 0$; $\Delta(x)\geqq 0$, $M(x)\geqq 0$. — The equation is transformed into an integral equation, regarding the delay term as a forcing function. Using the known results for the equation in which $M(x)\equiv 0$, the author derives a variety of results from the integral equation. Results concerning the oscillation of solutions are also given.

R. Bellman (Santa Monica, Calif.).

Markosyan, S. A. Sufficient conditions for the existence of several limiting cycles. Akad. Nauk Armyan. SSR. Dokl. 23 (1956), 153–159. (Russian. Armenian summary)

With $\dot{x}=dx/dt$, the system

$$\dot{x}=[f(x)+by]\Phi(x, y),\ \dot{y}=[ax+\phi(y)]F(x, y),$$

where $a, b=\pm 1$, is considered. Then the special case $\dot{x}=f(x)-y$, $\dot{y}=\phi(x)$ is considered. If $f(x)$ and the integral of $\phi(x)$ satisfy certain inequalities on a sequence of n intervals of the positive and negative x-axes, then it is shown that there are at least n limit cycles.

N. Levinson (Cambridge, Mass.).

★**Barocio, Samuel. On certain critical points of a differential system in the plane.** Contributions to the theory of nonlinear oscillations, vol. 3, pp. 127–135. Annals of Mathematics Studies, no. 36. Princeton University Press, Princeton, N. J., 1956. $4.00.

Using $[x, y]_p$ to denote a real convergent power series with least powers p, the author considers the real system $\dot{x}=ax+by+[x, y]_2$, $\dot{y}=cx+dy+[x, y]_2$, where the characteristic roots of the linear part are both zero but where at least one of a, b, c, d is not zero. In suitable coordinates he is then concerned with

$$\dot{x}=y+c(x),\ \dot{y}=-[y^2-2A(x)y+B(x)]E(x, y),$$

where $E(0, 0)=1$, $A=[x]_1$, $B=[x]_2$, $C=[x]_2$. This system he treats exhaustively as regards its qualitative behaviour. Sketches are made of the various phase-portraits.

N. Levinson (Cambridge, Mass.).

Sternberg, Shlomo. On differential equations on the torus. Amer. J. Math. **79** (1957), 397–402.

The system

$$\frac{dx}{dt}=f(x, y),\ \frac{dy}{dt}=g(x, y),$$

where f and g are of class C^n and period 1, defines a flow on the torus. Assuming an integral invariant $U(x, y)$ (and tacitly assuming $f^2+g^2>0$) it is shown that this system can be transformed into one of the form $dx/dt=F(x, y)$, $dy/dt=\gamma F(x, y)$. The constant γ is the ratio of the integral of Uf to that of Ug over the torus. In case F is analytic, the possibility of a further reduction to the form $dx/dt=A$, $dy/dt=B$, $B/A=\gamma$, by an analytic transformation of coordinates is shown to depend on the arithmetic nature of γ. It is sufficient that there exist positive constants K and h such that $|m-n\gamma|>Kh^n$ for all integers m and n. The latter result, as well as the former in the case of analytic flows, is due to Kolmogoroff [Dokl. Akad. Nauk SSSR (N.S.) **93** (1953), 763–766; MR **16**, 36]. In the present paper, the hypothesis $f^2+g^2>0$ is not stated, but must be assumed. Otherwise an example of a singular, ergodic, analytic flow on the torus given by the reviewer [Proc. Amer. Math. Soc. **4** (1953), 982–987; MR **15**, 730] would contradict the first theorem. In this connection the correction [J. Math. Soc. Japan **4** (1952), 338; MR **14**, 769] to a paper by Saitô cited by the author should also be noted.

J. C. Oxtoby (Bryn Mawr, Pa.).

Hahn, Wolfgang. Über Differential-Differenzengleichungen mit anomalen Lösungen. Math. Ann. **133** (1957), 251–255.

In a previous paper [Math. Ann. **131** (1956), 151–166; **132** (1956), 94; MR **17**, 1215; **18**, 402] the author gave a short proof of the known result that all the solutions of a differential-difference equation of linear form approach zero as the independent variable becomes infinite provided that all the zeros of the characteristic function are uniformly bounded away from the imaginary axis and lie in the left half-plane. In this paper he investigates the case in which the zeros are merely constrained to lie within the left half-plane without a uniform bound on the real parts, carrying through the discussion in detail for a particular equation where the asymptotic form of the characteristic roots can be determined.

R. Bellman.

Słowikowski, W. A generalisation of Mikusiński's operational calculus. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 643–647.

The author seeks a solution for the differential system

$$x^n(t)+A_1x^{n-1}(t)+\cdots+A_nx(t)=a(t)\ (-\infty<t<\infty),$$

$$x^j(0)=x_j\ (j=0, 1, \cdots, n-1).$$

Here X_0 is a linear topological space, locally convex and sequentially complete; X is the space of functions $y(t)$ which are continuous in t and valued in X_0; the A_i are given linear continuous endomorphisms of X_0; the x_j are given elements in X_0; the $a(t)$ is an element (given) in X and the solution $x=x(t)$ is required to be in X.

The classical method of successive approximations could be applied here to show that there is one and only one solution. However the author seeks to express the solution by means of an algebraic formula, using appropriate inversion operators. For this purpose he lets S denote the operator on X: $Sx=y$ means $x(t)=\int_0^t y(\tau)d\tau$. Then X is imbedded in a space $\hat{H}$ so that S can be extended to an automorphism of $\hat{H}$ [this employs a completion process described by the author in same Bull. **3** (1955), 3–6; MR **17**, 63]. Then the author assumes that the polynomial $\xi^n+A_1\xi^{n-1}+\cdots+A_n$ can be suitably factored as $(\xi-B_1)^{n_1}(\xi-B_2)^{n_2}\cdots(\xi-B_k)^{n_k}$; with this assumption, he obtains an algebraic expression, using inverses of operators $S-B_r$, to express the solution of the given differential system.

I. Halperin.

See also: Kazarinoff, p. 133; Gumowski, p. 139; Štraus, p.155; Klamkin, p. 181; Pisarenko, p. 195; Hecht and Mayer, p. 214.

Partial Differential Equations

Lions, J. L. Sur les problèmes aux limites du type dérivée oblique. Ann. of Math. (2) **64** (1956), 207–239.

Notazioni: Ω è un insieme aperto dello spazio euclideo R^n (in questa recensione considereremo, per semplicità, solo il caso di Ω limitato, tuttavia alcuni risultati dell'A. sussistono in ipotesi più generali per Ω); Γ è la frontiera di Ω di classe C^∞; $W^m(\Omega)$ lo spazio hilbertiano delle funzioni complesse dotate in Ω di derivate m-esime di quadrato sommabile; $\mathscr{D}(R^n)$ [$\mathscr{D}(\Omega)$, $\mathscr{D}(\Gamma)$] è la classe delle funzioni C^∞ a supporto compatto, contenute in R^n [Ω, Γ]; $C(\bar\Omega)$ è la restrizione a Ω delle funzioni di $\mathscr{D}(R^n)$; $\gamma_k(F)$ $(k=0, 1, \cdots, m-1)$ è la traccia su Γ della derivata k-esima $\partial^k F/\partial\nu^k$ secondo la normale ν a Γ di una funzione $F \in W^m(\Omega)$; $\gamma(F)$ è il vettore di componenti

$$(\gamma_0(F), \cdots, \gamma_{m-1}(F)).$$

$W_0{}^m(\Omega)$ è il sottospazio (completo) di $W^m(\Omega)$ definito dalla condizione $\gamma(F)=0$; $K^m(\Omega)$ è il complemento ortogonale di $W_0{}^m(\Omega)$ rispetto a $W^m(\Omega)$; U_k è il sottospazio (completo) di $W^m(\Omega)$ determinato dalla condizione $\gamma_k(u)=0$; V_k è il complemento ortogonale di U_k rispetto a $W^m(\Omega)$; p denota una n-pla di interi non negativi $p_1, \cdots, p_n$ e $|p|=p_1+p_2+\cdots+p_n$; D^p è la operazione di derivazione $\partial^{|p|}/\partial x_1{}^{p_1}\cdots\partial x_n{}^{p_n}$; g_{pq} una funzione misurabile e limitata in Ω; $(u, v)_g$ una forma sesquilineare (cioè lineare rispetto ad u e semilineare rispetto ad v) così definita in $W^m(\Omega)\times W^m(\Omega)$:

$$(u,v)_g = \sum_{|p|,|q|\le m} \int_\Omega g_{pq} D^p u \overline{D^q v}\, dx.$$

L'Autore definisce $\mathscr{H}$ come il sottospazio di $W^m(\Omega)$ costituito da tutte le funzioni u di $W^m(\Omega)$, tali che per ognuna di esse esiste una $f \in L^2(\Omega)$ tale che: $(f, \varphi)_{L^2(\Omega)} = (u, \varphi)_g$ per ogni $\varphi \in \mathscr{D}(\Gamma)$. Pone per definizione

$$\Lambda u \equiv \sum_{|p|,|q|\le m} (-1)^p D^p(g_{pq} D^q u) = f.$$

$\mathscr{H}$ è hilbertiano ponendo $\|u\|_{\mathscr{H}}{}^2 = \|u\|_{W^m} + \|\Lambda u\|_{L^2}$. Sia $(\mathscr{D}(\Gamma))^m$ lo spazio degli m-vettori $\mathbf{f}=(f_0, \cdots, f_{m-1})$ con $f_k \in \mathscr{D}(\Gamma)$. Si indica con $h(\mathbf{f})$ la funzione di $K^m(\Omega)$ (che esiste ed è unica) determinata dalla condizione: $\gamma[h(\mathbf{f})]=\mathbf{f}$. Per $\mathbf{f}$, $\mathbf{g} \in (\mathscr{D}(\Gamma))^m$ si pone: $(\mathbf{f}, \mathbf{g})_{H^m} = (h(\mathbf{f}), h(\mathbf{g}))_{W^m}$; H^m è lo spazio di Hilbert determinato per completamento di $(\mathscr{D}(\Gamma))^m$ rispetto a questo prodotto scalare. H^m risulta contenuto in $(L^2(\Gamma))^m$ (spazio degli m-vettori con componenti appartenenti a $L^2(\Gamma)$) ed ha una topologia più fine. Per ogni $u \in \mathscr{H}$ e $\mathbf{f} \in H^m$, l'A. considera la forma

$$(\Lambda u, h(\mathbf{f})) - (u, h(\mathbf{f}))_g.$$

Essa è un funzionale semi-lineare e continuo di $\mathbf{f}$. Detto H'^m il duale di H^m, esiste un elemento $T(u) \in H'^m$ tale che:

$$(1) \qquad (\Lambda u, v)_{L^2} - (u, v)_g = \langle T(u), \overline{\gamma(v)}\rangle, \; u \in \mathscr{H}, \; v \in W^m(\Omega).$$

$\langle\rangle$ denota la dualità fra H^m e H'^m. Tale relazione viene chiamata formola di Green. L'A. dichiara di non volersi servire della rappresentazione dei funzionali semilineari e continui in H^m, che porterebbe ad identificare H^m con H'^m. {Nota del recensore: È da ritenere ciò sia dovuto al fatto che detta rappresentazione identifica $T(u)$ con un m-vettore di H^m che in generale non coincide con lo m-vettore che interviene nella comune formola di Green}.

Con $h_k(f)$ si denota la (unica) funzione di V_k che verifica le condizioni

$$\gamma_0[h_k(f)]=0, \cdots, \gamma_k[h_k(f)]=f, \cdots, \gamma_{m-1}[h_k(f)]=0 \; (f \in \mathscr{D}(\Gamma)).$$

Posto: $(f, g)_{H_k{}^m} = (h_k(f), h_k(g))_{W^m}$, si definisce per completamento uno spazio di Hilbert $H_k{}^m$. Riesce $H_k{}^m \subset L^2(\Gamma)$ e $H_k{}^m$ ha una topologia più fine di $L^2(\Gamma)$. Sia B un operatore differenziale lineare a coefficienti di classe C^∞ in R^n. Esso sia di ordine $2m-1-k$. Posto: $B^\Gamma(F) = \gamma_0[B(F)]$, $(F \in \mathscr{D}(\Gamma))$, può esprimersi $B^\Gamma(F)$ per mezzo di derivazioni relative a parametri curvilinei scelti su Γ e di derivazioni rispetto a ν. Sia $m-1$ l'ordine di $B^\Gamma(F)$ rispetto a $\partial/\partial\nu$ su Γ (ordine di trasversalità di B rispetto a Γ). In tali ipotesi per B, l'A. dimostra che la forma sesquilinea $\int_\Gamma (B^\Gamma(F))\bar{f} ds$ è continua in $C(\bar\Omega)\times\mathscr{D}(\Gamma)$ assumendo in $C(\bar\Omega)$ la topologia indottavi da $W^m(\Omega)$ e in $\mathscr{D}(\Gamma)$ quella indottavi da $H_k{}^m$. Si ha allora $\int_\Gamma B^\Gamma(F)\bar{f} ds = \langle \mathscr{B}^\Gamma(F), \bar{f}\rangle_{H_k{}^m}$, denotando $\mathscr{B}^\Gamma(F)$ una trasformazione lineare continua di $W^m(\Omega)$ in $H_k{}'^m$ (duale di $H_k{}^m$) e $\langle\rangle_{H_k{}^m}$ la dualità fra $H_k{}^m$ e $H_k{}'^m$. L'A. assume $\mathscr{B}^\Gamma(F)$ come un prolungamento a $W^m(\Omega)$ di $B^\Gamma(F)$. Anche in questo caso l'A. non si serve dell'isomorfismo (canonico) di $H_k{}'^m$ con $H_k{}^m$. {Nota del recensore: Tale isomorfismo porterebbe ad identificare $\mathscr{B}^\Gamma(F)$ con una funzione di $H_k{}^m$ che per $F \in C(\bar\Omega)$ in generale non coincide con la funzione $B_\Gamma(F)$}. Sia B_k un operatore lineare di ordine $2m-1-k$ e $m-1$ volte trasversale a Γ. Si pone per $u, v \in V$: $((u, v)) = (u, v)_g + \sum_{k=0}^{m-1} \langle \mathscr{B}_k{}^\Gamma(u), \overline{\gamma_k(v)}\rangle_{H_k{}^m}$. Sia V un sottospazio completo di $W^m(\Omega)$ $(V \supset W_0{}^m(\Omega))$. Un teorema di Schwartz [Second colloque sur les équations aux dérivées partielles, Bruxelles, 1954, Thone, Liège, 1955, pp. 13–24; MR **17**, 745] e Lions [Acta Math. **94** (1955), 13–153; MR **17**, 745] assicura che, posto:

$$((u, v))_1 = \tfrac{1}{2}[((u, v)) + (\overline{(v, u)})],$$

se riesce $((v, v))_1 \ge a(\|v\|_{W^m})^2$ $(a>0)$ per ogni $v \in V$ [V-ellitticità della forma $((u, v))$], assegnato $f \in L^2(\Omega)$, esiste ed è unico $u \in V$ tale che: (2) $(f, v)_{L^2} = ((u, v))$ per ogni $v \in V$. Riesce inoltre: $\|u\|_{W^m} \le K\|f\|_{L^2}$ con K costante indipendente da u ed f. La scelta di V subordina vari tipi di problemi al contorno del tipo "derivata obliqua" per l'equazione $\Lambda u = f$. Ad esempio, per $V \equiv W^m(\Omega)$, la u che verifica le (2), verifica, per la (1), le condizioni al contorno con derivate oblique: $T_k(u) - B_k{}'(u) = 0$. {Nota del recensore: Applicando come esempio la teoria dell'A. al particolare problema piano

$$\frac{\partial^2 u}{\partial x_1{}^2} + \frac{\partial^2 u}{\partial x_2{}^2} + cu = f \text{ in } \Omega; \; \frac{\partial u}{\partial \nu} + q(s)\frac{\partial u}{\partial s} = 0 \text{ su } \Gamma$$

(c e q reali), la dimostrazione dell'esistenza e unicità di una soluzione viene ad aversi nel senso delle (2), quindi di tipo "debole", dopo aver dimostrata per le funzioni di $C(\bar\Omega)$ la diseguaglianza:

$$\int_\Omega (|\text{grad } v|^2 + c|v|^2) dx_1 dx_2 \ge \tfrac{1}{2}\int_\Gamma q \frac{\partial |v|^2}{\partial s} ds \ge a\int_\Omega [|\text{grad } v|^2 + |v|^2] dx_1 dx_2.$$

L'A. dimostra che la forma $((u, v))$ è V-ellittica se

$$\tfrac{1}{2}[(v, v)_g + \overline{(v, v)_g}] \ge a_1 \|v\|_{W^m} \; (a_1>0),$$

$$\sum_{k=0}^{m-1} [\langle \mathscr{B}_k{}^\Gamma(v)\gamma_k(\bar v)\rangle + \langle \mathscr{B}_k{}^\Gamma(v), \gamma_k(v)\rangle] \ge -2a_2(\|v\|_{W^m})^2 \quad (a_2>0, \; a_2<a_1).$$

Quest'ultima condizione si verifica, come egli osserva, se

gli operatori $\mathscr{B}_k{}^\Gamma$ sono di norma "assez petite". Relativamente al caso $m=1$, in opportune ipotesi per i coefficienti g_{pq}, l'A. afferma che è V-ellittica la forma

$$((u, v))_p=(u, v)_g+\langle\mathscr{B}_0{}^\Gamma u, \gamma_0(v)\rangle+p(\gamma_0(u), \gamma_0(v))_k$$

quando Re p è abbastanza grande (essendo

$$(f, f)_k=\iint_{\Gamma\times\Gamma} K(s, t)f(s)\overline{f(t)}dsdt$$

con $K(s, t)=\overline{K(t, s)}$ continuo in $\Gamma\times\Gamma$ e tale che

$$(f, f)_k\geqq a_3\int_\Gamma |f|^2 ds \ (a_3>0)$$

per $f\in\mathscr{D}(\Gamma)$). {Nota del recensore: Un nucleo $K(s, t)$ verificante le condizioni richieste dall'A. non esiste.} Il lavoro contiene le trattazione dei problemi di autovalori su Ω e su Γ e un cenno di applicazione ai problemi di propagazione per equazioni del secondo ordine.

G. Fichera (Roma).

Nirenberg, Louis. Uniqueness in Cauchy problems for differential equations with constant leading coefficients. Comm. Pure Appl. Math. **10** (1957), 89–105.

Pour un opérateur L d'ordre m dans R^n, la seule solution de $Lu=0$ qui s'annule avec ses dérivées jusqu'à l'ordre $m-1$ au voisinage d'un point x donné sur une surface à $n-1$ dimension passant par x, est-elle identiquement nulle en ce point? Le théorème est établi dans des domaines relativement spéciaux et pour des opérateurs qui s'avèrent assez généraux, quoique devant satisfaire à une inégalité compliquée. La démonstration est relativement élémentaire et ne fait plus appel aux développements en série de fonctions sphériques comme dans le travail de E. Heinz qui a servi de point de départ à la présente note [Nachr. Akad. Wiss. Göttingen. IIa. **1955**, 1–12; MR **17**, 626]. Elle repose sur une inégalité de L. Hörmander [Acta Math. **94** (1955), 161–248; MR **17**, 853] dont l'auteur donne une démonstration simple, due à Jack Schwartz. Le théorème établi contient comme cas particulier l'énoncé relatif à des fonctions continument dérivables, pourvues de dérivées secondes continues par sections et telles que

$$|\Delta u|^2\leqq C\cdot[|\text{grad } u|^2+|u|^2],$$

dans un ouvert connexe Ω: ces fonctions sont identiquement nulles dans Ω lorsqu'elles s'annulent au voisinage d'un seul point de Ω.

H. G. Garnir (Liège).

Haimovici, Mendel. Quelques propriétés des éléments intégraux d'un système de Pfaff du II^e genre. Rev. Math. Pures Appl. **1** (1956), no. 1, 23–32.

A translation from the Romanian of the article reviewed in MR **17**, 740.

Yablokov, V. A. Integration of first-order partial differential equations by the method of contact transformations. Kazan. Inst. Inžen.-Stroit. Neft. Promyš. Nauč. Trudy. **1953**, no. 1, 47–72. (Russian)

This article discusses the usual method of applying contact transformations to the integration of the equation $F(x, y, z, p, q)=0$, based on setting $z_1=0$, where by z_1 is meant the left side of the given equation. Here the functions x_1 and y_1, which are in involution with z_1 and with themselves, are considered as given.

The use of infinitesimal contact transformations is also considered. Here the characteristic function W must satisfy the relations $[W, F]-WF_z'=\omega(F)$ and $[W, F]=0$, where ω is an arbitrary function. This leads to the equation in F:

$$F_p'^3\frac{\partial}{\partial z}\left[\frac{F_x'+F_z'p}{F_p'}\right]=\frac{\partial}{\partial z}\left[\frac{F_y'+F_z'q}{F_q'}\right]=0.$$

Then it is possible to carry out an integration with respect to z, in the result of which there appear arbitrary functions of x, y, p, q, by specialization of which the desired integration can be brought to completion.

The article allows us to find more or less extensive classes of equations of the above form for which integrals can be found without quadratures. But it does not give any intrinsic criteria for saying when the methods under discussion are or are not applicable to a given equation. Examples are given which illustrate the results of the article. There are many misprints. At the end of page 58 an entire line is omitted: "determine the general integral of the equation $\Phi=0$."

A. D. Myškis (RŽMat 1954, no. 2146).

Auslander, Louis. An ideal theory for exterior differential equations. Ann. of Math. (2) **63** (1956), 527–534.

The objective of this paper is to apply the modern theory of Grassmann algebra to the study of the characteristic vector space, genus, and prolonged system of a system of exterior differential equations as defined by Cartan [Les systèmes différentiels extérieurs···, Hermann, Paris, 1945; MR **7**, 520]. In the first paragraphs the author develops some properties of the ideals of the Grassmann algebra $\Lambda(V^*)$ of V^*, the dual space to V. The two algebras $\Lambda(V)$, $\Lambda(V^*)$ are dually paired and graded by the degree of the element. Only ideals generated by homogeneous elements are considered. He gives, for example, a necessary and sufficient condition that a collection of homogeneous subspaces $Q_1, Q_2, \cdots, Q_N$ of degree $1, 2, \cdots, N$, $N=\dim\Lambda(V)$, should be the annihilators of an ideal in $\Lambda(V^*)$, namely for each i and each $z\in V^*$ we must have $Q_i \llcorner z\subset Q_{i-1}$. In this section of the paper he defines a minimal ideal M of an ideal A to be any ideal which (1) contains A, (2) is generated by its elements of degree one, and (3) contains no ideal with properties (1) and (2). This notion is applied to obtain a theorem that the characteristic vector space of an ideal A is contained in the intersection of the annihilators of the minimal ideals of A; in fact, if A is the intersection of its minimal ideals then the two spaces coincide. Also the two spaces coincide if A has genus greater than one and is generated by its elements of degree one and two (a frequent occurrence in exterior differential systems). Other theorems on genus and characteristic space are given and the paper concludes with a short paragraph on the prolonged system of an exterior differential system.

W. M. Boothby (Evanston, Ill.).

Lyubič, Yu. I. On the fundamental solutions of linear partial differential equations of elliptic type. Mat. Sb. N.S. **39**(81) (1956), 23–36. (Russian)

The purpose of this article is to set up criteria for the existence of a fundamental solution for the elliptic equation

$$Au=-\Delta u+c(x)u=\theta,$$

for $n=2, 3$. Let S be the closed n-dimensional sphere, let $D^2(S)$ be the class of functions twice, and $D(S)$ of functions once, continuously differentiable in S, and let $W(S)$ be the set of all functions $\theta\in D(S)$ for which the equation $Au=\theta$ has a solution in $D^2(S)$.

Theorem 1 (basic). For the existence in S of a funda-

mental solution it is necessary and sufficient that $W(S)$ be dense in $L^2(S)$.

The author also establishes the following connection between existence of fundamental solutions for the above equation and uniqueness of the solution of the Cauchy problem for the same equation.

Theorem 2. For the existence in S of a fundamental solution it is necessary and sufficient that in the class $D^2(S)$ the Cauchy problem

$$Au(x)=0,\ u|_\Sigma=\left.\frac{\partial u}{\partial \nu}\right|_\Sigma=0$$

has only the identically vanishing solution.

Some consequences of this theorem are deduced, which establish certain sufficient conditions for the existence of a fundamental solution of the given equation. The author refers to articles of Levi [Uspehi Mat. Nauk 8 (1940), 249–292] and Lopatinskiĭ [Ukrain. Mat. Ž. 3 (1951), 3–38; MR 16, 928] dealing with the same problem.

G. N. Agaev (RŽMat 1957, no. 3144).

Cordes, H. O. Über die eindeutige Bestimmtheit der Lösungen elliptischer Differentialgleichungen durch Anfangsvorgaben. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. **1956**, 239–258.

Let G be a region in E^n ($n\geqq 2$) with interior point ξ_0, u a solution in G of the general quasi-linear elliptic partial differential equation of the second order,

$$\sum_{i,k=1}^{n} a_{ik}\left(x, u, \frac{\partial u}{\partial x_j}\right)=f\left(x,u, \frac{\partial u}{\partial x_j}\right)$$

with a_{ij} twice continuously differentiable in its arguments, f satisfying a uniform Lipschitz condition in u and $\partial u/\partial x_j$. It is shown that if u has a zero of arbitrarily high order at ξ_0 — more precisely, if

$$u(x)|x-\xi_0|^{-l},\ \frac{\partial u}{\partial x_j}|x-\xi_0|^{-l},\ \frac{\partial^2 u}{\partial x_i \partial x_j}\ |x-\xi_0|^{-l}$$

are uniformly bounded near ξ_0 for each positive l — then u must be identically zero. Such a result, which includes the solution of the Cauchy problem as a special case, was established for $n=2$ by Carleman [C.R. Acad. Sci. Paris **197** (1933), 471–474] and for general n, extended to equation of the form $\Delta u=f(x, u, \partial u/\partial x_j)$ by C. Muller [Comm. Pure Appl. Math. **7** (1954), 505–515; MR **16**, 42], E. Heinz [Nachr. Akad. Wiss. Göttingen. IIa. **1955**, 1–12; MR **17**, 626] and Hartman and Wintner [Amer. J. Math. **77** (1955), 453–474; MR **17**, 855]. The result established in the present paper was first announced by N. Aronszajn [C. R. Acad. Sci. Paris **242** (1956), 723–725; MR **17**, 854]. The uniqueness of the Cauchy problem for the general equation above was also established by I. M. Landis [Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 640–643; MR **17**, 1212].

F. Browder (Paris).

Wolska, J. Problème aux limites à la dérivée tangentielle pour l'équation du type elliptique. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 563–567.

This paper considers the solution of the equation

$$\Delta u=F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

on a region D of the plane bounded by a Jordan curve C, with boundary condition $\partial u/\partial n+a(s)u=\Phi(s, u, \partial u/\partial s)$ on C. Assumptions: The direction tangential to C satisfies a Hölder condition with exponent γ, $|\Phi(s, u, t)-\Phi(s_1, u_1, t_1)| \leqq K[|s-s_1|^\alpha+|u-u_1|^\beta+|t-t_1|]$, F satisfies a Hölder condition in all variables, $|a(s)-a(s_1)|\leqq K_1|s-s_1|^\alpha$, $\alpha<\beta\leqq 1$, $\alpha<\gamma\leqq 1$. The author states an existence theorem for the solution of the problem provided that K, sup $|F|$, and sup $|\Phi|$ are sufficiently small. It is inclicated that the proof may be obtained from the Schauder fixed-point theorem by transforming the problem into a system of integral equations.

F. Browder (Paris).

Vekilov, Š. I. Mixed boundary problems for a set of piecewise-smooth harmonic functions. Akad. Nauk Azerbaĭdžan. SSR. Trudy Inst. Fiz. Mat. 4–5 (1952) 149–167. (Russian. Azerbaijanian summary)

Let D_1 and D_2 be (multiply-connected) regions, with D_1 bounded by smooth surfaces S, S_1, S_2, $\cdots$, S_n, and D_2 bounded by S, S_{n+1}, S_{n+2}, $\cdots$, S_{n+m+1}. Suppose that S surrounds S_1, S_2, $\cdots$, S_n, and that S_{n+m+1} surrounds $S, S_1, S_2, \cdots, S_{n+m}$. Define a set of n (unknown) functions U_1, U_2, $\cdots$, U_n by requiring that they be harmonic in D_1 and D_2 and satisfy the following conditions,

$$(dU_k/dn)=\sum_{j=1}^{N} a_{kj}^{(\nu)}U_j+f_k^{(\nu)} \text{ on } S_\nu,$$

$$K_1(dU_k/dn)_i\Big|_S=K_2(dU_k/dn)_e\Big|_S.$$

Here K_1, K_2 are constants, $a_{kj}^{(\nu)}$ and $f_k^{(\nu)}$ are continuous functions, the $a_{kj}^{(\nu)}$ satisfy certain inequalities, and it is required that the U_i be continuous across S. Then the U_k exist and are uniquely determined. The proof is obtained by representing the U_k as the potential due to single layers distributed over the surfaces S and S_ν, and hence obtaining Fredholm integral equations for the densities of these layers. If the boundary condition on S_{n+m+1} is replaced by one on the U_k not involving derivatives, then, in this case also, the U_k exist and are uniquely determined. In this case, as one would expect, Fredholm equations are obtained by replacing the single layer on S_{n+m+1} by a double layer.

R. B. Davis.

Vekilov, Š. I. On a plane problem concerning the Laplace equation. Akad. Nauk Azerbaĭdžan. SSR. Trudy Inst. Fiz. Mat. 6 (1953), 40–61. (Russian. Azerbaijanian summary)

Let g_1 be a connected plane region bounded by closed non-intersecting curves L_0, L_1, $\cdots$, L_m, σ, possessing continuous curvature. Let L_0 contain L_1, L_2, $\cdots$, L_m, σ in its interior. Let g_2 be a similar region bounded by σ, L_{m+1}, L_{m+2}, $\cdots$, L_{m+n}. Let σ contain L_{m+1}, L_{m+2}, $\cdots$, L_{m+n} in its interior. Let $g^{(i)}$ be the connected region bounded by L_i ($i=1, 2, \cdots, m+n$). The unknown harmonic function p is defined by $\Delta p=0$ in g_1 and g_2, $p=f(s)$ on $L_0+L_1+\cdots+L_{m+n}$, $k_1\ \partial p/\partial n|_{\sigma-}=k_2\ \partial p/\partial n|_{\sigma+}$ (a specified jump condition on the normal derivative at σ), $p|_{\sigma-}=p|_{\sigma+}$ (p is continuous across σ). Then p exists and is unique. (This problem arises in the diffusion of fluids through porous media.) The proof depends upon solving a modified problem, the essential feature of which is that the condition $p=f(s)$ is weakened to the condition $p=f(s)+a_k$ on L_k, where the constants a_k are to be determined along with the unknown p (except that a_0 is required to be 0), but an additional condition is superimposed, namely, that p be representable as the potential due to a single layer on σ and double layers on the curves L_j. This modified problem can readily be reduced to a pair of Fredholm integral equations [cf. N. I. Muscheli-švili, Prikl. Mat. Meh. (N.S.) **4** (1940), no. 4, 3–26; Soobšč. Gruzin. Fil. Akad. Nauk SSSR **1** (1940), 99–106; MR **3**,

151; 1, 314], and the usual Fredholm orthogonality conditions serve to determine the a_k uniquely, so that the solution of the modified problem exists and is unique.

The original problem can now be solved as follows. Uniqueness is a consequence of Green's formulas and a potential-theoretic line of argument. To prove existence, represent p in the form

$$p(P)=\pi^{-1}\int_\sigma \delta(q)\ln(r_q^{-1})d\sigma_q+$$
$$\pi^{-1}\int_L \mu(q)[d\ln(r_q^{-1})/dn_q]dl_q+\sum_{j=1}^{m+n}B_j\ln[r_j^{-1}(P, P_j)],$$

where P_j is a fixed point interior to $g^{(j)}$ (and therefore $P_j \notin \overline{g_1+g_2}$). This can be re-cast as a problem of the modified type, the constants B_j depend linearly on the constants a_j of the modified problem, and the determinant of the coefficients in this linear relation cannot be zero. In the original problem, the requirement $a_j=0$ $(j=1, 2, \cdots, m+n)$ serves to determine a unique set of constants B_j and consequently a unique function p.

R. B. Davis (Syracuse, N.Y.).

Reina, Ida. Sulla soluzione di un problema di Neumann relativo a un dominio rettangolare. Rend. Mat. e Appl. (5) **15** (1956), 366–384 (1957).

Si consideri il problema di Neumann

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+k^2u=0 \text{ in } R,\quad \frac{\partial u}{\partial \nu}=\psi \text{ su } \mathscr{F}R,$$

dove R è un rettangolo del piano (x, y), $\mathscr{F}R$ è la sua frontiera, ν la normale interna; nell'ipotesi che k^2 non sia autovalore per tale problema e che la funzione ψ coincida su ogni lato di $\mathscr{F}R$ con una funzione di una variabile hölderiana e a variazione limitata, è dimostrata l'esistenza (e l'unicità) di una soluzione continua con le derivate prime in $R+\mathscr{F}R$.

E. Magenes (Genova).

Dubovickiĭ, A. Ya. On some properties of solutions of an elliptic system of linear partial differential equations. Vologod. Ped. Inst. Uč. Zap. **11** (1953), 159–178. (Russian)

New proofs are given for some known properties of the elliptic system

$$\frac{\partial u(x, y)}{\partial x}=a(x, y)\frac{\partial v(x, y)}{\partial y},$$
$$\frac{\partial u(x, y)}{\partial y}=-b(x, y)\frac{\partial v(x, y)}{\partial x},$$

where $0<m\leq a$, $b\leq M$. The central point of the article is the proof, for the solution of this system, of the following theorem (analogous to the theorem of Picard in a somewhat weaker formulation): if two functions $u(x, y)$, $v(x, y)$ are twice differentiable and everywhere on the xy plane satisfy the second of the above equations, then they assume all pairs of values u, v with the possible exception of a set of measure zero.

E. M. Landis (RŽMat 1954, no. 2582).

Blondel, Jean-Marie. Comportement des solutions d'une équation linéaire du second ordre, au voisinage d'une singularité d'un coefficient. C. R. Acad. Sci. Paris **242** (1956), 981–983.

The equation

$$(ax+by)^n\frac{\partial^2 z}{\partial x\partial y}=A(x, y)z,$$

where A is bounded between two positive numbers, can be reduced to an equation of similar form, where $ax+by$ is replaced by $x+y$ when $ab>0$, by $x-y$ when $ab<0$, or by x when $ab=0$. For the first two cases, given Cauchy data are taken along the line $x\pm y=1$, and the behavior of the solution z is considered on the line $x\pm y=\varepsilon$, as ε tends to 0. For the third case, bounded values of z are given on the lines $y=0$ and $x=1$, and the behavior of the solution z is considered on the line $x=\varepsilon$, as ε tends to 0. The results, given without proof, are different for the three cases and for various values of n.

D. L. Bernstein.

Ciliberto, Carlo. Sul problema di Darboux per l'equazione $s=f(x, y, z, p, q)$. Rend. Accad. Sci. Fis. Mat. Napoli (4) **22** (1955), 221–225 (1956).

Conditions sufficient to ensure the existence of a solution of the problem: $s=f(x, y, z, p, q)$, $z(x, 0)=z(0, y)=0$ are obtained, which are of slightly different character from those given by the author in a previous paper [Ricerche Mat. **4** (1955), 15–29; MR **17**, 621]. The proof depends upon showing that the problem is equivalent to finding a fixed point for the functional transformation $z'=T(z)$, where T is a rather complicated operator, and then showing that this transformation is completely continuous and takes a certain closed convex set into part of itself, so that the theorem of Brouwer can be applied.

D. L. Bernstein (Los Angeles, Calif.).

Corduneanu, C. La dépendance des solutions des équations hyperboliques par rapport aux coefficients et aux données sur les caractéristiques. Rev. Math. Pures Appl. **1** (1956), no. 1, 41–44.

A translation from the Romanian of the article reviewed in MR **17**, 1214.

Smirnov, M. M. Functionally invariant solutions of the wave equation. Leningrad. Gos. Univ. Uč. Zap. **135**. Ser. Mat. Nauk **21** (1950), 127–202. (Russian)

Oleĭnik, O. A.; and Ventcel', T. D. The first boundary problem and the Cauchy problem for quasi-linear equations of parabolic type. Mat. Sb. N.S. **41**(83) (1957), 105–128. (Russian)

This paper contains detailed proofs of results announced in an earlier note [Dokl. Akad. Nauk SSSR (N.S.) **97** (1954), 605–608; MR **16**, 259] and also some extensions, notably to systems of equations.

R. Finn.

Tingley, Arnold J. On a generalization of the Poisson formula for the solution of the heat flow equation. Proc. Amer. Math. Soc. **7** (1956), 846–851.

By modifications and extensions of the method used by R. H. Cameron [Ann. of Math. (2) **59** (1954), 434–462; MR **15**, 799] for the one-dimensional heat equation, the author solves the boundary value problem of the n-dimensional heat equation: $\Delta G+\theta(t, \bar\xi)G-a\partial G/\partial t=0$ $(0<t<t_0\leq\infty$, $a=$ positive constant), $\lim_{t\downarrow 0}G(t, \bar\xi)=\sigma(\bar\xi)$ for almost all $\bar\xi=(\xi_1, \cdots, \xi_n)$.

Under suitable assumptions on $\theta(t, \bar\xi)$ and $\sigma(\bar\xi)$ of the nature of exponential growths as $\sum_{i=1}^n \xi_i^2\to\infty$, the equation is solved by an n-fold Wiener integral over the product space C^n of the Wiener space C:

$$G(t, \bar\xi)=\int_{C^n}\exp\Big\{ta^{-1}\int_0^1\theta[t(1-s), 2(ta^{-1})^{\frac12}x_1(s)+\xi_1, \cdots,$$
$$2(ta^{-1})^{\frac12}x_n(s)+\xi_n]ds\Big\}$$
$$\times\,\sigma[2(ta^{-1})^{\frac12}x_1(1)+\xi_1, \cdots, 2(ta^{-1})^{\frac12}x_n(1)+\xi_n]d_wx_1\cdots d_wx_n.$$

K. Yosida (Tokyo).

Èĭdel'man, S. D. On regular and parabolic systems of partial differential equations. Uspehi Mat. Nauk (N.S.) 12 (1957), no. 1(73), 254–258. (Russian)

Article condensé (souvent réduit à de simples indications bibliographiques) sur les équations d'évolution, à coefficients fonctions du temps ou non mais indépendants de la variable d'espace; ces équations sont résolues par transformation de Fourier en x, en utilisant les nombreux espaces fonctionnels introduits par Gel'fand et Silov [Uspehi Mat. Nauk (N.S.) **8** (1953), no. 6 (58), 3–54; Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 1065–1068; MR **15**, 867; **17**, 267] et étudiés ou utilisés par de nombreux auteurs.

A la bibliographie on peut ajouter l'intéressante note résumée çi-après. *J. L. Lions* (Nancy).

Borok, V. M. On a characteristic property of parabolic systems. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 903–906. (Russian)

On considère le système différentiel

$$(1)\qquad \frac{\partial}{\partial t}u=Pi\frac{\partial}{\partial x}u\ \ (x\in R,\ t>0),$$

où $u=(u_1,\cdots,u_N)$, P=matrice à coefficients constants. On suppose le système parabolique au sens de Šilov [Uspehi Mat. Nauk (N.S.) **10** (1955), no. 4(66), 89–100; MR **17**, 495].

Alors le problème de Cauchy pour (1) est bien posé lorsque la donnée initiale $u_0(x)$ appartient à un espace S_p défini par des conditions de croissance à l'infini. L'A. montre a) que la solution du problème de Cauchy est indéfiniment différentiable dans $t>0$; b) que réciproquement si le problème de Cauchy est bien posé (dans un sens raisonnable) alors (1) est parabolique au sens de Šilov. *J. L. Lions* (Nancy).

Vâlcovici, V. Sur le théorème des valeurs extrêmes (TVE). Rev. Math. Pures Appl. **1** (1956), no. 1, 33–40.

A translation from the Romanian of the article reviewed in MR **17**, 491.

Bakievič, N. I. Some boundary problems for equations of mixed type in a strip and in a half plane. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 793–796. (Russian)

The author studies the equation

$$(1)\quad y^m a(y)\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\left\{-\frac{n}{y}+b(y)\right\}\frac{\partial u}{\partial y}+c(y)u=0,$$

where $a(y)$, $b(y)$, $c(y)$ are analytic functions of y, $a(y)>0$, and $m=n=2k-1$ or else $m=-1$, $n=k-1$, $k=1, 2, 3, \cdots$. This equation is of mixed elliptic-hyperbolic type in each of the regions

$$(2a)\qquad -\infty<x<+\infty,\ -\alpha<y<\beta \qquad (\alpha>0,\ \beta>0)$$

$$(2b)\qquad -\infty<x<+\infty,\ -\infty<y<\beta \qquad (\beta>0)$$

$$(2c)\qquad -\infty<x<+\infty,\ -\alpha<y<+\infty \quad (\alpha>0).$$

For an appropriate one of these regions, the author proves the existence of a solution of (1) satisfying one of the boundary conditions

$$(3a)\qquad u(x,y)_{y=\beta}=F_1(x)\quad (-\infty<x<+\infty,\ \beta\neq\infty)$$

or

$$(3b)\qquad u(x,y)_{y=-\alpha}=F_2(x)\quad (-\infty<x<+\infty,\ \alpha\neq\infty)$$

and the additional condition

$$(4)\qquad \lim_{y\to+0}y^{-n-1}u(x,y)=\lim_{y\to-0}y^{-n-1}u(x,y)$$

provided the function $F_1(x)$ (or $F_2(x)$) is suitably restricted. (Special attention is given to the case of periodic initial conditions.) The proof is based on a two-sided Laplace transformation in the variable x. Some details were not clear to the reviewer. *R. Finn.*

Redheffer, R. M. A Sturmian theorem for partial differential equations. Proc. Amer. Math. Soc. **8** (1957), 458–462.

Let the nonlinear differential operator $L(w)$ be defined by

$$L(w)=\sum a_{ij}(x_k,w_k/w)w_{ij},$$

where $w_k=\partial w/\partial x_k$, $w_{ij}=\partial^2 w/\partial x_i\partial x_j$ and $a_{ij}(x_k,w_k/w)=a_{ij}(x_1,\cdots,x_n,w_1/w,\cdots,w_n/w)$. In a region R, the function w is called an admissible partial solution of

$$(*)\qquad L(w)/w\geqq\phi(x_k,w_r/w)\ \ (x_1,\cdots,x_n)\in R,$$

if w_i is differentiable in R for each i and if (*) holds at every point of R where $w\neq0$. If in R there exists a sequence of compact subsets $R_n\subset R_{n+1}\subset R$, $\lim R_n=R$, such that on the boundary of R_n one has $\sup w\leqq m+1/n$, then we say $\inf\sup w\leqq m$ at the boundary. The main result in the paper is the following Sturmian theorem: Let $u\not\equiv0$ be an admissible partial solution of (*) in R such that $\inf\sup|u|\leqq0$ at the boundary. Suppose that the matrix (a_{ij}) is positive semidefinite, and also suppose that $\phi^*(x_k,s_k)<\phi(x_k,s_k)$ for all relative values of the $2n$ arguments. Then every admissible partial solution of $L(v)/v\leqq\phi^*(x_k,v_k/v)$ satisfies $\inf|v|=0$ in R.

F. G. Dressel (Durham, N.C.).

Walter, Wolfgang. Ganze Lösungen der Differentialgleichung $\Delta^p u=f(u)$. Math. Z. **67** (1957), 32–37.

The equation of the title is considered with $p\geqq2$, u being a function of n independent variables. The author shows that if $f(s)$ is a continuous positive function in $(-\infty,+\infty)$ with $f(s)>s^{1+\alpha}$, $\alpha>0$, for $s\geqq s_0>0$, no entire solution (i.e., $2p$-continuously differentiable in R^n) exists if $n=2$; if furthermore $f(s)\geqq\delta>0$, the result remains true for $n>2$. A counter-example is given to show that the last assumption is essential. The same conclusion holds for the inequality $\Delta^p u\geqq f(u)$ and consequently for equations $\Delta^p u=f(u)+F(x,u,u_{x_i},\cdots)$ with $F\geqq0$.

J. L. Massera (Montevideo).

Burgers, J. M. Statistical problems connected with the solution of a simple non-linear partial differential equation. I, II, III. Nederl. Akad. Wetensch. Proc. Ser. B. **57** (1954), 403–413, 414–424, 425–433.

I. L'A. résume d'abord les résultats qu'il a obtenus dans des publications antérieures [même Proc. Ser. B. **57** (1954), 45–56, 57–66, 67–72, 159–169; MR **15**, 961]. Il s'agit des solutions „turbulentes" de l'équation

$$\partial v/\partial t+v\partial v/\partial y=\nu\partial^2 v/\partial y^2,$$

dont on sait écrire la solution générale en fonction de $v(t_0,y)=-a(y)$. Lorsque ν est très petit, on écrit cette solution à l'aide de l'artifice suivant. On construit la „courbe de sommation" $s(\eta)=\int_0^\eta a(\xi)d\xi$. Une parabole $s_p(\eta)=(\eta-y)^2/[2(t-t_0)]$ glisse sur la courbe de sommation, la touchant au point d'abscisse ξ. On a $v(t,y)=-a(y+\xi)$. Lorsqu'il y a double contact, $v(t,y)$ subit un saut. Entre deux sauts, v est à peu près linéaire en y. Pour étudier les propriétés statistiques de v, on doit donc partir de celles de a, puis étudier celles de la courbe de sommation s. En écrivant que $s=\lim_{\varepsilon\to0}\varepsilon\sum_{i=1}^{\eta/\varepsilon}a_i$, et en supposant les a_i

aléatoires, nuls en moyenne, indépendants si $\varepsilon|i-j|<\theta$, et statistiquement homogènes, on peut calculer les moments de s. Pour les grandes valeurs de η, ils sont ceux d'une loi normale d'écart type $(2J\eta)^{\frac{1}{2}}$, où

$$J=\int_0^\theta R(\zeta)d\zeta,\quad R(\zeta)=\overline{a(\xi)a(\xi+\zeta)}.$$

La densité $\psi(\eta, s)$ de cette loi vérifie une équation de diffusion $\partial\psi/\partial\eta=J\partial^2\psi/\partial s^2$ et définit la probabilité pour qu'une courbe de sommation, partant de l'origine, passe entre les points (η, s) et $(\eta, s+ds)$. On se propose ensuite de déterminer la probabilité pour qu'une telle courbe touche, sans la recouper, une parabole considérée comme „barrière absorbante". Ce problème ne peut d'ailleurs être abordé qu'avec une certaine approximation.

II. Soit $E(\alpha)$ la probabilité pour qu'une courbe partant du point $(0, \Delta-\Delta_1)$ atteigne $+\infty$ sans rencontrer (avec l'approximation Δ), la „barrière d'absorption" représenté par la parabole $s=(\alpha\eta+\eta^2)/2t$. $E'(\alpha)$ est la probabilité pour qu'une courbe partant du point $(0, \Delta-\Delta_1)$ ait au moins un sommet entre les deux paraboles définies par $\alpha+d\alpha$ et α, sans rencontrer la première. Si $Y(l, \alpha)/\delta$ représente la densité de celles de ces courbes ayant leur premier sommet entre les abscisses l et $l+dl$, on a

$$E(\alpha)=\frac{1}{\delta}\int_0^\infty dl\int_{-\infty}^S dS'Y(l,S'),\ S=\alpha l+\frac{l^2}{2t}.$$

On définit la densité $\Psi(l, S)$, au point $(l, S+\Delta-\Delta_1)$, des trajectoires issues du point $(0, \Delta-\Delta_1)$, et touchant la parabole sans la traverser. On vérifie que

$$Y(l, S)=\Psi(l, S)E(\alpha+l/t).$$

On peut alors calculer la probabilité pour qu'une courbe touche une parabole donnée en 0 et en A sans jamais la traverser. On en déduit les divers moments de la distance aléatoire l entre deux contacts successifs, puis la densité de probabilité de la distance des axes de deux paraboles successives, d'où la densité de probabilité de la distance de deux points de discontinuité successifs de $v(y, t)$. On obtient enfin une expression pour la quantité

$$\partial R(\zeta)/\partial\zeta=\partial[\overline{a(\xi)a(\xi+\zeta)}]/\partial\zeta.$$

III. Le modèle de turbulence étudié satisfait à une loi de similitude. Si l'on pose

$$\eta=\eta_* J^{1/3}t^{2/3},\ l=l_* J^{1/3}t^{2/3},\ \delta=\delta_* J^{1/3}t^{2/3},$$
$$s=s_* J^{2/3}t^{1/3},\ S=S_* J^{2/3}t^{1/3},\ \Delta=\Delta_* J^{2/3}t^{1/3},$$

t et J disparaissent des équations. l_* croit comme $t^{2/3}$. On montre ensuite que $\Psi(l, S)=\Phi(l)\exp(-S^2/4Jl)$. La fonction $\Phi(l)$ ne dépend pas de S, on peut en donner des expressions approchées. Si en particulier la „barrière absorbante" est une parabole très plate, on trouve que

$$\Phi=(\Delta_1^2/2(\pi J^3l^3)^{1/2})(1-\tfrac{1}{4}\pi^{1/2}l_*^{3/2}+\cdots).$$

En ce qui concerne $E(\alpha)$, on en indique d'abord une borne supérieure et une borne inférieure. La relation

$$\frac{dE}{d\alpha}=\frac{1}{\delta}\int_0^\infty dl\cdot l\Psi(l,S)E\left(\alpha+\frac{l}{t}\right)$$

permet ensuite d'écrire le développement de $E(\alpha)$ suivant les puissances négatives de α. Pour terminer, l'A. donne des valeurs numériques approximatives des principales grandeurs dont l'étude mathématique a été faite. Il obtient en particulier les valeurs de $E(\alpha)/\Delta_1$, et il trouve que $l=0{,}92J^{1/3}t^{2/3}$. *J. Bass* [Zbl. 58, 124].

See also: Halilov, p. 178; Aržanyh, p. 193; Vorovič, p. 193; Babič, p. 197; Barenblatt, p. 208; Selig and Fieber, p. 213; White, p. 213.

Difference Equations, Functional Equations

Fort, Tomlinson. Limits of the characteristic values for certain boundary-value problems associated with difference equations. J. Math. Phys. 35 (1957), 401–407.

A rather general second order difference equation containing a parameter is considered. The values of the parameter lie in a set of characteristic values associated with a boundary value problem. Qualitative features of these characteristic values are obtained. For instance, they are all positive. It is shown that not all of them tend to zero or infinity as another parameter in the equation tends to infinity.

Special cases of the equation considered appear in chain and net problems. *E. Pinney* (Berkeley, Calif.).

Tietze, Heinrich. Zwei Sonderfälle eines Grenzwertproblems. Math. Nachr. 13 (1955), 283–287.

Discussion of the behavior of the nth derivative of an interpolating polynomial of order n. As the author himself points out in the article reviewed below, his results follow from the known properties of the divided differences. *G. G. Lorentz* (Detroit, Mich.).

Tietze, Heinrich. Bemerkung zu meiner Note: Zwei Sonderfälle eines Grenzwertproblems. Math. Nachr. 15 (1956), 173.

The author states that the limit problem discussed in his earlier article (reviewed above) was already solved by T. J. Stieltjes [Verslagen Akad. Wet. Amsterdam, Afd. Natuurk., 2 Ser. 17 (1882), 239–254].

Parker, R. V. A method of summing rational integral functions. Math. Gaz. 41 (1957), 134–136.

By repeated use of the identity

$$\sum_{r=1}^{p}\left\{\sum_{r=1}^{n}(-1)^{r+1}\binom{n}{r}k^{n-r}\right\}=p^n,$$

sums of the form $\sum_{k=1}^{p} R(k)$ are found, where $R(k)$ is a rational integral function of k. *D. Moskovitz.*

Sumner, D. B. A generalized averaging operator. Canad. J. Math. 8 (1956), 437–446.

In questo lavoro, dato un numero reale positivo λ arbitrario, mediante le posizioni

(1) $\nabla_h^\lambda f(z)=$

$$\sum_{p=0}^{N+1}\binom{N+1}{p}\int_{c-i\infty}^{c+i\infty}f(z+ph-hw)\frac{\Gamma(w)\Gamma(\mu-w)}{2\pi i\Gamma(\mu)2^\lambda}\,dw$$

$(c>0$, N intero e $\lambda\leq N<\lambda+1$, $\mu=N+1-\lambda)$,

(2) $\nabla_h^{-\lambda}f(z)=$

$$\frac{2^\lambda}{2\pi i}\int_{c-i\infty}^{c+i\infty}f(z-hw)\frac{\Gamma(w)\Gamma(\lambda-w)}{\Gamma(\lambda)}\,dw\quad(0<c<\lambda),$$

si definiscono le potenze ad esponente reale qualunque $\pm\lambda$ per l'operatore di media (di passo h)

$$\nabla_h f(z)=\tfrac{1}{2}[f(z+h)+f(z)],$$

e si dimostrano i teoremi seguenti: (i) se $f(z)$ è una funzione intera $=O(\text{esp}\ \kappa|z|)$ per $|z|\to\infty$, con $\kappa|h|<\pi$, allora

esistono gli integrali a secondo membro delle (1) e (2), le funzioni $\nabla_h^\lambda f(z)$ e $\nabla_h^{-\lambda} f(z)$ risultano anche essere intere ed $=O(\text{esp } \kappa|z|)$ per $|z|\to\infty$, ed è $\lim_{h\to 0} \nabla_h^\lambda f(z)=f(z)$; (ii) se $f(z)$ verifica le ipotesi del teorema (i), α, β e λ sono tre numeri reali positivi, allora si ha

$$\nabla_h^1 f(z)=\nabla_h f(z),\ \nabla_h^\alpha \nabla_h^\beta f(z)=\nabla_h^{\alpha+\beta} f(z),\ \nabla_h^{-\lambda}\nabla_h^\lambda f(z)=f(z).$$

Questi risultati inducono poi l'A. a generalizzare la nozione di „polinomio di Eulero" [cf. Milne-Thomson, Proc. London Math. Soc. (2) **35** (1933), 514–522; e quella di "numero C" di Nörlund Vorlesungen über Differenzenrechnung, Springer, Berlin, 1924, p. 27], finora introdotte solo in corrispondenza dell'operatore ∇_h^N (N intero), e di studiarne alcune proprietà formali.

F. Bertolini (Roma).

Fenyő, I. Über eine Lösungsmethode gewisser Funktionalgleichungen. Acta Math. Acad. Sci. Hungar. **7** (1956), 383–396. (Russian summary)

The main idea of this interesting, clearly written and important paper can be described as follows: Functional equations are often solved by reducing them to differential equations. This method has the advantage of generality, but it has also a disadvantage, namely it yields only the (in some cases higher order) differentiable solutions of the original functional equation. Though there are several results and methods known [see, e.g., M. Kac, Comment. Math. Helv. **9** (1937), 170–171] on the differentiability of integrable solutions of certain functional equations and types of functional equations, the method of the present author is more general. It is based upon the theory of distributions. The original functional equation is considered as an equation for distributions. As distributions are always differentiable several times, the equation can be reduced without difficulty to a differential equation of distributions. This is solved and the author proves afterwards that the resulting distribution is a function. Here he makes use of the integrability-condition. But of course the author's method can also be considered as a procedure of solving distribution-equations. After an introduction on the theory of distributions the author applies his method to the equations $f(x+y)=\sum_{k=1}^n g_k(x)h_k(y)$, $f(ax+by+c)=g(x)+h(y)$, $f(x+y)+f(x-y)=f(x)g(y)$, etc. [see, e.g., T. Levi-Civita, Atti Acad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) **22** (1913), 2° semestre, 181–183; J. Aczél, Comment. Math. Helv. **21** (1948), 247–252; MR **9**, 514; W. H. Wilson, Bull. Amer. Math. Soc. **26** (1920), 300–312], and to equivalent and special types.

J. Aczél (Debrecen).

Fort, M. K., Jr. Research problem number 22. Math. Student **24** (1956), 189–191 (1957).

The author considers the functional equation

$$f(x)=\max(g(x)+f(ax),\ h(x)+f(bx)),$$

typical of those appearing in the theory of dynamic programming, and establishes the existence and uniqueness of the solution by means of the method of successive approximations. [For more general results see Trans. Amer. Math. Soc. **80** (1955), 51–71; MR **17**, 632.]

R. Bellman (Santa Monica, Calif.).

See also: van der Corput, p. 136; Hahn, p. 145.

Integral and Integrodifferential Equations

Ivanov, V. V. Approximate solution of singular integral equations when the integral is not taken along a closed contour. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 933–936. (Russian)

Using the fundamental theorem of a previous paper [same Dokl. (N.S.) **110** (1956), 15–18; MR **18**, 906] and results of Al'per [Izv. Akad. Nauk SSSR Ser. Mat. **19** (1955), 423–444; MR **17**, 729], the author first extends the results of his previous paper to the case where the contour of integration is a smooth simple closed curve. This is then employed to treat the case where the contour of integration consists of a finite number of non-intersecting smooth arcs.

J. B. Diaz (College Park, Md.).

See also: Zukhovitsky, p. 154; Stépaniuk, p. 155; Elliott, p. 185; Glauberman, p. 191; Yakovleva, p. 197; Bychawski, p. 198; Bartels and Downing, p. 200; Mills, p. 218.

Calculus of Variations

Križanič, France. Linear functionals on Banach space and the fundamental lemma of the calculus of variations. Acad. Serbe Sci. Publ. Inst. Math. **10** (1956), 59–70.

The author presents abstract versions of generalizations of the fundamental lemma of the calculus of variations. For example, theorems of Razmadzé [Math. Ann. **84** (1921), 115–116] and Haar [J. Reine Angew. Math. **149** (1919), 1–18] appear thus: Let A_k ($k=1, \cdots, n$) denote linear operators with domains D_k dense in a Banach space X, and with bounded inverses which commute; let $x_1^*, \cdots, x_n^*$ be bounded linear functionals on X such that $\sum x_k^*(A_k x)=0$ for $x\in \bigcap_k D_k$. Then there exist functionals $y_1^*, \cdots, y_n^*$ such that $\sum y_k=0$, and for each k, $(A_1^*\cdots A_{k-1}^* A_{k+1}^*\cdots A_n^*)y_k^*=x_k^*$.

M. Jerison.

See also: Finzi, p. 198.

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Groups

★**Pontrjagin, L. S. Topologische Gruppen. I.** B. G. Teubner Verlagsgesellschaft, Leipzig, 1957. 263 pp. DM 15.00.

A translation of the edition reviewed in MR **17**, 171. The eleven chapters of the original Russian have now been divided into two volumes. The present first volume contains Chapters 1 through 5. The second volume is promised soon.

Lie Groups and Algebras

Berezin, F. A.; and Gel'fand, I. M. Some remarks on the theory of spherical functions on symmetric Riemannian manifolds. Trudy Moskov. Mat. Obšč. **5** (1956), 311–351. (Russian)

Let G be a semi-simple Lie group, and let $x\to x^*$ be an involutory anti-automorphism of G such that the subgroup G_0 of all x with $x^*=x^{-1}$ is compact and such that every x in G may be written in the form uh, where $u\in G_0$

and $h^*=h$. Then every $G_0:G_0$ double coset is invariant under *, and it follows that those members of the group algebra $L^1(G)$ which are constant on the $G_0:G_0$ double cosets form a commutative subalgebra R of $L^1(G)$. Now let L be an arbitrary irreducible unitary representation of G whose restriction to G_0 contains the identity as a direct summand. It follows from the commutativity of R that this restriction contains the identity just once. Thus if ϕ is a unit vector in the space of L such that $L_\xi(\phi)=\phi$ for all $\xi \in G_0$, then the function $x\to(L_x(\phi), \phi)$ is uniquely determined by L. It is a continuous function, constant on the $G_0:G_0$ double cosets, which we shall denote by ϕ_L. The functions of the form ϕ_L are called zonal spherical functions. If ϕ_L is a zonal spherical function, then the map $f\to\int \phi_L(x)f(x)dx$ is a homomorphism of R onto the complex numbers, and every such homomorphism may be so obtained.

These known facts and notions and others related to them are recalled in the introductory section of the paper. In the second section the law of multiplication in the ring R is studied. Considering first the case in which G is the direct product with itself of the group of all $n\times n$ unitary unimodular matrices and G_0 is the diagonal subgroup, the members of R are considered as functions on the set D of $G_0:G_0$ double cosets and the multiplication law thrown into the form $(f_1f_2)(t)=\iint f_1(t_1)f_2(t_2)a(t_1, t_2, t)dt_1dt_2$, where t, t_1, and t_2 are in D and a is an explicitly described function on $D\times D\times D$. There are relations between the function a, the zonal spherical functions and certain differential operators, and these are explored in detail. In this case (and in general when G is the direct product with itself of a compact group and G_0 is the diagonal subgroup) the zonal spherical functions are in a natural one-to-one correspondence with the characters of the original compact group. Next it is briefly indicated how these results may be generalized to the case in which the unitary unimodular group is replaced by an arbitrary compact semi-simple Lie group. Further indications then treat the closely parallel facts which hold in the case in which G is a complex semi-simple Lie group and G_0 is a maximal compact subgroup of G. By way of application a theorem on the proper values of sums and products of matrices is proved. (For a statement of this theorem see the review of a paper of Lidskiĭ [Dokl. Akad. Nauk SSSR (N.S.) **75** (1950), 769–772; MR **12**, 581] in which another proof is given.)

In the third section it is shown that the zonal spherical functions satisfy the functional equation $\phi_L(x_1)\phi_L(x_2)=\int_{G_0}\phi_L(x_1ux_2)du$ and that the right-hand side may be interpreted as the average value of ϕ_L over a "sphere" of "radius" $G_0x_1G_0$ with center a "distance" $G_0x_2G_0$ from the origin G_0. The $G_0:G_0$ double cosets are in natural one-to-one correspondence with the equivalence classes of pairs of points in the homogeneous space G/G_0 under the action of G and in this sense can be regarded as generalized distances. This functional equation is examined in detail in the special case in which G/G_0 is the n-dimensional sphere.

The results of section 4 are also presented in terms of generalized distances. Let $V_{t,x}$ denote the operator which takes a function f on G/G_0 into its mean value on a sphere with center at x and "radius" t. Let $\Delta^1, \Delta^2, \cdots, \Delta^n$ denote the generalized Laplace operators for G/G_0 introduced in earlier articles [Gel'fand, ibid. **70** (1950), 5–8; MR **11**, 498; Gel'fand and Cetlin, ibid. **71** (1950), 825–828; MR **12**, 9]. Then if $\psi(t, x)=V_{t,x}f$ for some function f, we have $\Delta_x{}^k\psi(t, x)=\Delta_t{}^k\psi(t, x)$, where on the left side t is held fixed and on the right side x is held fixed and ψ as a function of t is regarded as a function of a point in G/G_0 a distance t from G_0. In addition it is shown that $V_{t,x}$ may be expressed as a function of the operators $\Delta^1, \Delta^2, \cdots, \Delta^n$, the function being constructed from the zonal spherical functions.

Let G be a compact semi-simple Lie group and let Ξ denote the algebra of all complex-valued functions ξ on the equivalence classes of irreducible representations such that $\|\xi\|=\sum_L |\xi(L)|\dim (L)<\infty$ and where

$$\xi_1\xi_2(L)=\sum_{L_1, L_2}\xi_1(L_1)\xi_2(L_2)G(L_1, L_2, L).$$

Here $G(L_1, L_2, L)$ is the number of times that L is contained in the reduction of the Kronecker product of L_1 and L_2. Section five is chiefly devoted to the structure and properties of this algebra (defined in a different but equivalent manner) and especially to the analogies that exist between it and the algebra R of section two. Here the function G plays the role of the function a in section two and has many analogous properties. Again a special case is treated in detail and generalizations more briefly indicated. At the end of the section some of the results of section 2 on complex semi-simple groups are generalized to the real case and to certain non-semi-simple groups. *G. W. Mackey* (Cambridge, Mass.).

Vilenkin, N. Ya. On the theory of associated spherical functions. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 742–744. (Russian)

Let H be a (suitably restricted) compact subgroup of the Lie group G. Let T be an irreducible unitary representation of G and let S be an irreducible constituent of the restriction T^H of T to H. Let n be the multiplicity of occurrence of S in T^H. To each pair T, S the author makes correspond an $n\times n$ matrix-valued function on G. These are his associated spherical functions. Their traces are the spherical functions considered by Godement [Trans. Amer. Math. Soc. **73** (1952), 496–556; MR **14**, 620] in a more general setting. When S is the identity representation and $n=1$ they reduce to the spherical functions considered by Gel'fand [Dokl. Akad. Nauk SSSR (N.S.) **70** (1950), 5–8; MR **11**, 498] and by Berezin and Gel'fand [see the paper reviewed above]. The author announces generalizations to his matrix-valued functions of some of the results of Gel'fand, Berezin and Godement.

G. W. Mackey (Cambridge, Mass.).

Vilenkin, N. Ya. Bessel functions and representations of the group of Euclidean motions. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 3(69), 69–112. (Russian)

Let $\mathfrak{G}_2$ be the group of all rigid motions of the Euclidean plane, realized as the group of all matrices

$$g(r, \varphi, \alpha)=\begin{pmatrix} \cos\alpha & -\sin\alpha & r\cos\varphi \\ \sin\alpha & \cos\alpha & r\sin\varphi \\ 0 & 0 & 1 \end{pmatrix},$$

$0\leq r<\infty$, $0\leq\varphi<2\pi$, $0\leq\alpha<2\pi$. For $g=g(r, \varphi, \alpha)\in\mathfrak{G}_2$ and R real, let $M_g{}^{(R)}$ be the operator on $L_2(0, 2\pi)$ defined by $M_g{}^{(R)}f(\theta)=\exp[iRr\cos(\theta-\varphi)]f(\theta-\alpha)$. The mapping

$$g\to M_g{}^{(R)}$$

is a continuous, irreducible, unitary representation of $\mathfrak{G}_2$. Let $f_n(\theta)=e^{in\theta}$. Then

$$(M_g{}^{(R)}f_m, f_k)=i^{m-k}\exp[im(\phi-\alpha)-ik\phi]J_{m-k}(Rr),$$

where J_s is the Bessel function of (integral) order s.

From these facts, the author deduces a large number of known relations involving Bessel functions, by reasonably simple arguments. As he points out, the appearance of Bessel functions in irreducible unitary representations of $\mathfrak{G}_2$ would be a handy tool in finding new relations involving Bessel functions.

The author next takes up certain reducible unitary representations of $\mathfrak{G}_2$. He first discusses Kronecker products of representations $M_g^{(R)}$ and $M_g^{(R')}$, which are defined as operators on L_2 for the torus. He then considers the "quasi-regular" representation of $\mathfrak{G}_2$, by operators on $L_2(R^2)$. Consider $g \in \mathfrak{G}_2$ as a mapping of R^2 onto itself. For $f \in L_2(R^2)$, let $T_g f(x, y) = f(g^{-1}(x, y))$. Then T_g is a reducible unitary representation of $\mathfrak{G}_2$. It is shown to be a continuous direct sum of the irreducible representations $M_g^{(R)}$ $(R>0)$. The regular representation of $\mathfrak{G}_2$ by operators on $L_2(\mathfrak{G}_2)$ is also broken up into irreducible unitary components.

The author next takes up the trace of $M_g^{(R)}$, giving a reasonably convincing argument that it should be defined as $J_0(Rr)\delta(\alpha)$, where δ is Dirac's delta function. He also gives a heuristic argument relating Bessel functions to Jacobi polynomials, by making $\mathfrak{G}_2$ appear as a sort of limit of the 3-dimensional rotation group O_3.

Finally, the author classifies all continuous irreducible unitary representations of $\mathfrak{G}_2$ that satisfy a certain auxiliary condition. He does not mention the problem of finding all continuous irreducible unitary representations of $\mathfrak{G}_2$. The arguments used here are almost identical with and are apparently motivated by the standard Lie algebra construction of the irreducible representations of O_3.

The notion of studying special functions by means of group representations has a long history, going back to E. Cartan [Rend. Circ. Mat. Palermo **53** (1929), 217–252] and H. Weyl [Ann. of Math. (2) **35** (1934), 486–499]. It has been extensively developed by many writers, for example Gel'fand and his school [e.g., Gel'fand and Šapiro, Uspehi Mat. Nauk (N.S.) **7** (1952), no. 1(47), 3–117; MR **13**, 911; see also the two papers reviewed above] and by Godement [Trans. Amer. Math. Soc. **73** (1952), 496–556; MR **14**, 620]. *E. Hewitt* (Seattle, Wash.).

See also: Nomizu, p. 168.

Topological Vector Spaces

Słowikowski, W. A topologisation of the conjugate space of a locally convex linear space. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 113–115, XI. (Russian summary)

Let X' denote the conjugate space of a locally convex linear topological space X. Examples are known where X' is an F-space but is not a t-space (espace tonnelé), and is not bornological in the strong topology [J. A. Dieudonné, Bull. Amer. Math. Soc. **59** (1953), 495–512; MR **15**, 963]. The author introduces a new topology for X', in terms of which he obtains several new results, including the following. He shows that X' becomes bornological and is a t-space. Moreover, if X is bornological it can be isomorphically imbedded in X''. The new topology is defined as follows. First, a family $\mathscr{R}$ of subsets of X will be called bounded if for every neighborhood U of the origin there is a real number $\lambda>0$ and an element $A \in \mathscr{R}$ such that $\lambda A \subset U$. Now consider the bounded families $\mathscr{R}$ of non-empty, symmetric, convex subsets of X which contain with each pair A, B an element $C \subset A \cap B$. For each such $\mathscr{R}$, define the pseudo-norm $|x'|_{\mathscr{R}} = \inf_{A \in \mathscr{R}} (\sup_{x \in A} x'(x))$. The author's topology is that which is induced in X' by these pseudo-norms. *D. H. Hyers* (Los Angeles, Calif.).

Vasilach, Serge. Sur le produit de composition des fonctions et distributions à support dans $R_+{}^n$, $n>1$. C. R. Acad. Sci. Paris **243** (1956), 1708–1711.

The author has extended to certain n-dimensional cases the reviewer's theorem that if the convolution (resultant) of two functions vanishes, then one or other of the functions must vanish in certain regions. In a previous note [same C. R. 1591–1593; MR **18**, 491] the case where the two functions involved are identical was considered. In this note the theorem is extended to the case of any two functions. *E. C. Titchmarsh* (Oxford).

Altman, M. Invariant subspaces of completely continuous operators in locally convex linear topological spaces. Studia Math. **15** (1956), 129–130.

The author extends the theorem of N. Aronszajn and K. T. Smith [Ann. of Math. (2) **60** (1954), 345–350; MR **16**, 488] about existence of invariant subspaces for completely continuous operators in Banach spaces to completely continuous operators in locally convex spaces. The proof uses the theorem for Banach spaces.
N. Aronszajn (Lawrence, Kansas).

Berezanskiĭ, Yu. M.; and Kreĭn, S. G. Hypercomplex systems with continual basis. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 147–152. (Russian)

A survey of recent results by one or both of the authors, together with examples [see Dokl. Akad. Nauk SSSR (N.S.) **72** (1950), 5–8, 237–240, 825–828; **81** (1951), 329–332, 493–496; **85** (1952), 9–12; **91** (1953), 1245–1248; Ukrain. Mat. Ž. **3** (1951), 184–204, 412–432; Mat. Sb. N.S. **32(74)** (1953), 157–194; MR **12**, 188, 189; **13**, 952, 953; **14**, 162; **15**, 797; **14**, 944, 884, 746]. *R. A. Good.*

See also: Sargent, p. 126; Pełczyński, p. 135; Il'in, p. 136; Slugin, p. 140; Słowikowski, p. 145; Štraus, p. 155; Stépaniuk, p. 155; Elliott and Feller, p. 185; Sunčeleev, p. 192; Parasyuk, p. 221.

Banach Spaces, Banach Algebras, Hilbert Spaces

Zukhovitsky, S. I. On a minimum problem in certain spaces of number sequences. Dopovidi Akad. Nauk Ukrain. RSR **1957**, 3–7. (Ukrainian. Russian and English summaries)

Two theorems are stated without proofs. The first theorem is an extension of a result given by the author earlier [Kiïv. Derž. Univ. Mat. Zb. No. 2, 169 (1948)]. The second one is an extension of a recent result of W. W. Rogosinski [Math. Z. **63** (1955), 97–108; MR **17**, 273].

Theorem 1. Let c be the Banach space of all convergent sequences of real numbers; let $x_1, x_2, \cdots, x_n$ be n linearly independent elements in c; and let $c_1, c_2, \cdots, c_n$ be n given real numbers. Then the following statements hold: (1) If a linear functional (i) $f(x) = \sum_{k=0}^{\infty} f_k \xi_k$ in c satisfies the conditions (ii) $f(x_i) = c_i$ $(i=1, 2, \cdots, n)$, and if $\|f\| = \sum_{k=0}^{\infty} |f_k| = M$, then there exists at least one linear functional in c, say $f^0(x) = \sum_{k=0}^{\infty} f_k{}^0 \xi_k$, which satisfies condition (ii), has the same norm M, and is such that at most $n+1$ coordinates $f_k{}^0$ are different from zero. (2) Among the linear functionals in c satisfying condition

(ii) and having the minimal norm there is at least one which has at most n of its coordinates different from zero. (3) The estimates $n+1$ and n in (1) and (2) cannot be improved.

Theorem 2 states that if linear functionals of the form (i) are considered not in c but in a larger space m of all bounded sequences of real numbers, then statement (1) of Theorem 1 remains true; statement (2) also remains true with the necessary additional restriction that a functional in m exists which satisfies condition (ii) and has the form (i) and the minimal norm.

These theorems can be interpreted as concerning solutions of a system (iii) $\sum_{k=0}^{\infty} \xi_{ik} f_k = c_i$ $(i=1, 2, \cdots, n)$ of n linear equations with an infinite number of unknowns. To every solution of the system (iii) with $\sum_{k=0}^{\infty} |f_k| = M$ there corresponds a solution $\{f_k^0\}$, with the same norm, in which not more than $n+1$ unknowns are different from zero. If the system (iii), however, has minimal solutions, then at least one of these will have not more than n unknowns different from zero. *H. P. Thielman.*

Fréchet, Maurice. Le problème de l'existence d'un extremum local d'une fonctionnelle. Ann. Sci. Ecole Norm. Sup. (3) **73** (1956), 93–120.

In the author's (generalized) sense, a real-valued function $F(z)$, defined in the Banach space Z, has the kth variations $(k=1, \cdots, n)$ $F_n(y, u)$ for $y \in Z$, $u \in Z$, if, for real t, the expression

$$t^{-n}\{F(y+tu)-F(y)-\sum_{1}^{n} F_k t^k/k!\}$$

is a function of y, u, t which, for constant (u, y), tends to 0 with t. The object of the paper is to study, in terms of these variations, conditions analogous to those concerning maxima and minima in elementary calculus. The so-called doubtful case is also treated, but the author's treatment of it is stated to be a first contribution, not achieving the sharpness of the method of von Dantscher, which he quotes from Valiron's "Cours d'analyse", t. 1, 1re éd. [Masson, Paris, 1942, p. 258; MR **7**, 283]. *L. C. Young.*

Widom, Harold. Nonisomorphic approximately finite factors. Proc. Amer. Math. Soc. **8** (1957), 537–540.

The author previously generalized Von Neumann's definition of approximate finiteness (a.f.) for separable factors of type II_1 to factors whose density character in the trace norm is a cardinal $>\aleph_0$ [Trans. Amer. Math. Soc. **83** (1956), 170–178; MR **18**, 322]. He did this for each cardinal in two ways, a.f. (A) and a.f. (B). It was not yet clear whether the difference was real or only apparent. But it was known that there was only one a.f. (B) factor of a given density character. Now there are constructed $|\omega|+1$ nonisomorphic factors with density character $\aleph_\omega$, for any cardinal $\aleph_\omega > \aleph_0$, and all of which are a.f. (A). The factors constructed are all group algebras of discrete groups. *J. Feldman* (Berkeley, Calif.).

Daleckiĭ, Yu. L. Integration and differentiation of functions of Hermitian operators depending on a parameter. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 182–186. (Russian)

This is a short account of work published in full in the paper reviewed in MR **18**, 914. *J. L. B. Cooper.*

Maslov, V. P. Perturbation theory for the transition from discrete spectrum to continuous spectrum. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 267–270. (Russian)

Let μ_n and μ be measures defined on the real line such that $\lim_{n\to\infty} \mu_n(\Delta)=\mu(\Delta)$ for every interval Δ. A sequence of functions $\{f_n\}$ is said to converge to f in the mean according to the measures $\{\mu_n\}$, if

$$\lim_{n\to\infty} \int (f_n-f)^2 d\mu_n = 0.$$

Let $\{A_n\}$ be a sequence of self-adjoint operators with a common domain which is dense in a Hilbert space, and let A be the closure of $\lim_{n\to\infty} A_n$. Let the spectrum of each A_n be discrete. The main result of this paper deals with the convergence of the eigenfunctions of A_n to the generalized eigenfunctions of A. Here the convergence considered is the mean convergence according to the spectral measures, and the generalized eigenfunctions of A are taken in the sense of I. M. Gel'fand and A. G. Kostyučenko [same Dokl. (N.S.) **103** (1955), 349–352; MR **17**, 388]. Application to the Schrödinger equation is discussed. *Ky Fan* (Oak Ridge, Tenn.).

Štraus, A. V. A formula for generalized resolvents of a differential operator of even order. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 773–776. (Russian)

The author considers a formally self-adjoint ordinary differential operator L of order $2n$ with real coefficients on an arbitrary open interval (a, b). Associated with L is a certain operator T_0 in the Hilbert space $L^2(a, b)$ which is minimal, in the sense that it is the smallest closed symmetric operator containing in its domain all sufficiently differentiable functions which vanish outside compact subsets of (a, b). The main result stated by the author is that every generalized resolvent of T_0 is an integral operator of Carleman type. The indications given as to the proof show that the author uses very heavily the assumption of real coefficients. In this case, if c is any point interior to (a, b), the number m^- $[m^+]$ of linearly independent solutions of $Ly=ly$ for $\Im\, l\neq 0$ in $L^2(a, c)$ $[L^2(c, b)]$ is independent of l, and the deficiency index m of T_0 is given by $m=m^-+m^+-2n$. Using this the author indicates how the kernel of the generalized resolvent can be broken up into three parts corresponding to the solutions in $L^2(a, c)$, $L^2(c, b)$ and $L^2(a, b)$. All this explicit information is not available in the case of an L of arbitrary order with complex coefficients. The reviewer [Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 638–642; MR **18**, 915], after seeing A. V. Štraus' earlier work on the second order equation with real coefficients on the half-line [Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 201–220; MR **17**, 620], showed how to obtain all generalized resolvents and resolutions of the identity (with expansion theorem) for an L of arbitrary order. *E. A. Coddington* (Princeton, N.J.).

Stépaniuk, K. L. Quelques généralisations du principe du point stationnaire. Ukrain. Mat. Ž. **9** (1957), 105–110. (Russian. French summary)

The author proves the following generalization of Schauder's fixed-point theorem: Let A be a completely continuous operator defined on a set $M=M_1-\sum_{i=1}^{m} x_i$, where M_1 is an infinite-dimensional bounded subset of a Banach space and $\sum_{i=1}^{m} x_i$ is a finite subset of M_1. If $AM\subset\bar{M}_1$, there exists either 1) a fixed point of A or 2) a sequence $\{y_n\}$, $y_n \in M$, $\lim y_n = y_0$, such that $\lim Ay_n = y_0$. The infinite dimensionality of M_1 is necessary. A special case of the theorem is applied to the integral equation

$$\varphi(x)=\lambda\int_a^b K(x, t) f[t, \varphi(t)]dt+F(x),$$

with 1) $K(x, t)$ uniformly continuous on $a \leqq x, t \leqq b$, $F(x)$ continuous on $a \leqq x \leqq b$ and 2) $f(x, y)$ bounded and uniformly continuous on $a \leqq x \leqq b$, $|y| \leqq 2 \max \{|F(x)|; a \leqq x \leqq b\}$ except for a countable set P of curves $y=p(x)$, $p(x)$ an arbitrary polynomial. Theorem: For all sufficiently small real λ there exists either 1) a continuous solution of the integral equation or 2) a sequence $\{\varphi_n(x)\}$ of functions, continuous on $a \leqq x \leqq b$, uniformly convergent to a continuous function $\varphi_0(x)$, and such that

$$\lambda \int_a^b K(x, t) f[t, \varphi_n(t)] dt + F(x)$$

converges uniformly to $\varphi_0(x)$. If $\varphi_0(x)$ is analytic and not in P, it is the solution of the integral equation.

H. Komm (Troy, N.Y.).

See also: Marcus and Moyls, p. 114; Krasnosel'skiĭ and Kreĭn, p. 140; Vorovič, p. 144; Križanič, p. 152; Livšic, p. 216; Heisenberg, p. 220.

TOPOLOGY

General Topology

Massaro, Giliana Pannoli. Constructions à propos du problème de sélections. C. R. Acad. Sci. Paris **245** (1957), 294–297.

The author gives several examples illustrating the general problem of selections treated in a previous paper [C. R. Acad. Sci. Paris **244** (1957), 153–155; MR **18**, 813]. Given a "continuous" real-valued bounded relation (multivalent function) C, with domain an interval $[a, b]$, such that the image set of each x in $[a, b]$ is closed, a selection through (x, y) is defined to be a univalent function f included in C, which is defined and continuous in a neighborhood of x, and satisfies $f(x)=y$. A point is said to be inaccessible if there is no selection through it. A typical result is that, for an arbitrary closed everywhere-discontinuous subset K of $[a, b]$, there is a C such that only the points (x, y) with $x \in K$, $0<y<1$, are inaccessible.

E. Mendelson (Cambridge, Mass.).

Pomilio, Isabella. Gli assiomi di separazione in una classe di spazi topologici. Rend. Mat. e Appl. (5) **13** (1955), 391–405.

L'A. étudie les relations entre les axiomes topologiques susceptible d'être vérifiés dans un espace topologique (au sens de M. Fréchet) S, et les axiomes vérifiés dans l'espace topologique conjugué de S (au sens de L. Geymonat [Rend. Mat. e Appl. (5) **12** (1953), 360–366; MR **16**, 387]; on sait, d'après ce dernier auteur, que le conjugué du conjugué de S est S). — Sont particulièrement etudiés à ce point de vue les divers axiomes de séparation T_1, T_2, T_3, T_4 qui caractérisent les espaces accessibles, de Hausdorff, réguliers, et normaux. — Le bord d'un ensemble dans un espace topologique S devient l'intérieur de cet ensemble dans l'espace conjugué de S. — Conditions nécessaires et suffisantes pour que le conjugué d'un espace topologique soit un espace (V) (au sens de M. Fréchet), respectivement ne soit pas un espace (V).

A. Appert.

Ciarrapico, Lucia. Sulla permutabilità di due trasformazioni della contiguità negli spazi topologici. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 267–272.

Soit S un espace topologique (au sens de M. Fréchet) où la fermeture d'un ensemble quelconque E de points est désignée par $\bar{E}$. Soit $\dot{S}$ l'espace topologique conjugué de S au sens de L. Geymonat. Soit $\hat{S}$ l'espace topologique, dit agrégé à S, qui a les mêmes points que S et où la fermeture d'un ensemble quelconque E de points est, par définition, $\hat{E}=\bigcup_{e \subset E} \bar{e}$ [cf. Appert, Mathematica, Cluj **11** (1935), 229–246, p. 245]. — L'A. démontre que la condition nécessaire et suffisante pour que $\hat{\dot{S}}=\dot{\hat{S}}$ est que S soit un espace (V) (au sens de M. Fréchet) dont le conjugué soit aussi un espace (V). — Etude de l'itération alternative des opérations de conjugaison * et d'agrégation ^ appliquées à S.

A. Appert (Angers).

Ribeiro de Albuquerque, José. Connectedness properties in abstract spaces. Univ. Lisboa. Revista Fac. Ci. A. (2) **5** (1955–1956), 5–62. (Portuguese)

This paper studies the connectedness properties in topological spaces in the sense of Fréchet. Various generalizations of results of Fréchet and Appert are obtained. The "fixed objective" (to quote the author) is to solve the two problems proposed by A. Appert and Ky Fan [Espaces topologiques intermédiaires $\cdots$, Hermann, Paris, 1951, p. 112; MR **13**, 54]. The proof of Theorem 64, which is basic in the attempt to solve Fan's problems, appears to the reviewer to be in error. More specifically, the proof for S_4 given on p. 49 is incorrect. For if we let $I=\{a, b\}$, $\mathfrak{F}=\{\{a\}, \{b\}, \{a, b\}\}$, then $S^*=\{(\{a\}, \{b\}), (\{b\}, \{a\}), (\{a\}, \{a, b\}), (\{a, b\}, \{a\}), (\{b\}, \{a, b\}), (\{a, b\}, \{b\})\}$, and S_4 is clearly invalid (otherwise we would have $(\{a, b\}), \{a, b\}) \in S^*$, which is not the case).

Hing Tong (New York, N.Y.).

Nagata, Jun-iti. On a relation between dimension and metrization. Proc. Japan Acad. **32** (1956), 237–240.

Let R be a metrizable topological space. Let $\dim R$ denote the Lebesgue or Urysohn-Menger dimension of R [for the equivalence of these, see Katětov, Czechoslovak Mat. Ž. **2(77)** (1952), 333–368; MR **15**, 815]. Theorem: The inequality $\dim R \leqq n$ obtains if and only if R admits a metric ρ, compatible with its topology, satisfying the following condition. For all $\varepsilon>0$, $x \in R$, and $y_1, \cdots, y_{n+2} \in R$, the inequalities

$$\rho(S_{\varepsilon/2}(x), y_i)<\varepsilon \quad (i=1, 2, \cdots, n+2)$$

imply that $\rho(y_i, y_j)<\varepsilon$ for some i, j with $i \neq j$.

The proof is complicated. The $\varepsilon/2$ appearing in the statement of the theorem can be replaced by any positive function $\phi(\varepsilon)$. Corollary: R is 0-dimensional if and only if it admits a metric ρ for which $\rho(x, z) \leqq \max[\rho(x, y), \rho(y, z)]$. This corollary has also been proved by de Groot [Proc. Amer. Math. Soc. **7** (1956), 948–953; MR **18**, 325].

E. Hewitt (Seattle, Wash.).

Isbell, J. R. Zero-dimensional spaces. Tôhoku Math. J. (2) **7** (1955), 1–8.

Let uX denote a uniform space. The least n such that the uniformity u contains arbitrarily fine coverings of order n is called the large dimension (l.d.) of uX; the l.d. of fuX (fu consists of all finite coverings from u) is called the uniform dimension (u.d.) of uX. The invariance under completion is proved for l.d. and u.d.; $\dim X=\dim \beta X$ X normal) follows as a corollary. It is proved that uX is) zero-dimensional (u.d.) if and only if the completion of fuX is homeomorphic to the space of maximal ideals of the algebra of uniformly continuous $f \in T^X$, T being a two-element field. The second section concerns the inter-

relations of different kinds of zero-dimensionality and some other related properties; an example is given where a completely regular X has an open base of open-closed sets, whereas $\dim \beta X > 0$. The concluding section answers, mainly by counterexamples, some questions concerning homogeneity, etc., raised in a previous paper by the author [Duke Math. J. **20** (1953), 321–329; MR **14**, 1001].

M. Katětov (Prague).

Dowker, C. H. Local dimension of normal spaces. Quart. J. Math. Oxford Ser. (2) **6** (1955), 101–120.

For any topological space X, let $\dim X$ denote the (Lebesgue) covering dimension of X, and let $\operatorname{ind} X$ [$\operatorname{Ind} X$] denote the (Menger-Urysohn) inductive dimension of X defined in terms of the boundaries of neighborhoods of points [closed subsets] of X. Loc $\dim X \leq n$ [loc $\operatorname{Ind} X \leq n$] if n is the least integer such that every point of X has a neighborhood U such that $\dim \bar{U} \leq n$ [$\operatorname{Ind} \bar{U} \leq n$]; ind X is already a local concept.

It is shown that for any normal regular space X, one has $\operatorname{ind} X \leq \operatorname{loc} \operatorname{Ind} X \leq \operatorname{Ind} X$, $\operatorname{loc} \dim X \leq \operatorname{loc} \operatorname{Ind} X$, $\operatorname{loc} \dim X \leq \dim X \leq \operatorname{Ind} X$. Moreover, examples are given of normal Hausdorff spaces, which show that no other relations between ind, loc dim, loc Ind, dim, and Ind hold for all normal regular spaces. The author shows, however, that if X is normal, and either paracompact, or the union of two paracompact subsets one of which is closed, or the union of a sequence of closed paracompact subsets, then $\operatorname{loc} \dim X = \dim X$. [The first of these latter theorems was obtained independently by K. Nagami, Nagoya Math. J. **8** (1955), 69–70; MR **16**, 946.] The subset theorem, that $\dim A \leq \dim X$ for $A \subset X$ is established for X totally normal [this concept was introduced by the author, Quart. J. Math. Oxford Ser. (2) **4** (1953), 267–281; MR **16**, 157], thereby generalizing Čech's result [Časopis Pěst. Mat. Fys. **62** (1933), 277–291] that the subset theorem holds for X perfectly normal. Čech's problem on whether the subset theorem holds for X completely normal is reduced to the problem of deciding if $\operatorname{loc} \dim X = \dim X$ for X completely normal. Many other theorems, both of the subset-type and of the kind that relate the various local and global dimensions, are proved.

Two of the examples given (M and P) are of normal Hausdorff spaces such that $\operatorname{ind} X = 0$, but $\operatorname{Ind} X = 1$. Since, for normal Hausdorff spaces, $\operatorname{Ind} X = \operatorname{Ind} \beta X$ (where βX denotes the Stone-Čech compactification of X), they answer a question posed in the paper reviewed below. Another example of such a (not necessarily normal) space is given independently in the paper reviewed above.

M. Henriksen (Lafayette, Ind.).

Banaschewski, Bernhard. Über nulldimensionale Räume. Math. Nachr. **13** (1955), 129–140.

By a 0-dimensional space, the author means one with a basis consisting of open and closed sets. A uniform structure P on a completely regular Hausdorff space is called non-archimedean if for each entourage $U \in P$, we have $\overset{2}{U} = U$. A. F. Monna [Nederl. Akad. Wetensch., Proc. **53**, 470–481, 625–637 (1950); MR **12**, 41] has shown that a completely regular Hausdorff space admits a non-archimedean uniform structure if and only if it is 0-dimensional. The present author, exploiting the fact that the relation defined by $(x, y) \in U$ is an equivalence relation, obtains shorter proofs of many of Monna's theorems, and in addition derives some new results. The main theorems of the latter kind are mentioned below. E always denotes a 0-dimensional Hausdorff space.

E is a dense subspace of a compact 0-dimensional space ζE such that any other compact 0-dimensional space containing E as a dense subspace is a continuous image of ζE under a mapping that keeps E pointwise fixed. Moreover, ζE is the space of components (in the quotient-space topology) of the Stone-Čech compactification βE of E. If E is normal, $\zeta E = \beta E$ if and only if E has Lebesgue dimension 0 (the author calls it Čech dimension), or if and only if the space of bounded continuous real-valued functions on E whose range is finite is dense in the space of all bounded continuous real-valued functions on E. This last result seems to be well-known. The author states that he knows of no examples of spaces E for which $\zeta E \neq \beta E$. {Two such examples (M and P) of normal spaces are given by C. H. Dowker in the paper reviewed above, and another such example (not known to be normal) by J. R. Isbell in the paper reviewed second above.} He also shows that every locally compact metric E admits a non-archimedean metric, thereby solving a special case of a problem of Monna [op. cit., Problem 2].

{Finally, we remark that some of the theorems cited above overlap with results announced independently by P. J. McCarthy [Bull. Amer. Math. Soc. **61**, 182 (1955)] and J. R. Isbell [op. cit.].}

M. Henriksen.

Nagata, Jun-iti. A theorem for metrizability of a topological space. Proc. Japan Acad. **33** (1957), 128–130.

The author obtains the following necessary and sufficient condition for metrizability, which seems to be an easy consequence of all other known conditions of this sort, and which has the further virtues of simplicity and obvious necessity: A T_1-space is metrizable if and only if every x in X has (not necessarily open) neighborhoods $U_n(x)$ and $S_n(x)$ ($n=1, 2, \cdots$), with $\{U(x)\}_{n=1}^{\infty}$ a base for the neighborhoods of x, such that $S_n(y) \cap S_n(x) \neq \emptyset$ implies $y \in U_n(x)$, and $y \in S_n(x)$ implies $S_n(y) \subset U_n(x)$. {Reviewer's note: The above statement is slightly simpler than the author's, but obviously equivalent to it.}

E. Michael (Seattle, Wash.).

Papić, Pavle. Sur les espaces pseudo-distanciés complets. Glasnik Mat.-Fiz. Astr. Ser. II. **11** (1956), 135–142. (Serbo-Croatian summary)

Soit (D_α) la classe des espaces pseudo-distanciés correspondant à l'ordinal ω_α [cf. Kurepa, C. R. Acad. Sci. Paris **198** (1934), 1563–1565; Publ. Math. Univ. Belgrade **5** (1936), 124–132]; D_0 est la classe des espaces métriques. Si $\alpha > 0$, chaque D_α-espace est définissable par un tableau ramifié B vérifiant $\gamma B = \omega_\alpha$ et par un écart δ vérifiant $\delta(a, b) < \xi$ et $\delta(b, c) < \xi \Rightarrow \delta(a, c) < \xi$ [cf. Doss, C. R. Acad. Sci. Paris **223** (1946), 14–16; MR **8**, 48; Papić, Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **9** (1954), 197–216, 217–228; MR **17**, 515, 516]. Un tel écart est par exemple celui-ci: $\delta(a, a) = \omega_\alpha$, $\delta(a, b) = \xi$ $(a \neq b)$, ξ étant le premier indice ν tel que $R_\nu B$ contienne deux termes contenant respectivement a et b; soit δB cet écart. Un espace D_α, $\alpha > 0$, est complet relativement à δB si et seulement si l'intersection des éléments de chaque ω_α-chaîne $\subseteq B$ est non vide (Th. 1). Chaque $E \in (D_\alpha)$, $\alpha > 0$, est isométriquement contenu dans un espace E' complet de la classe (D_α) dans lequel E est partout dense (Th. 4). Pour l'espace Q des nombres rationnels l'A. indique deux distances telles que le complété relatif $\bar{Q}$ soit isomorphe respectivement à l'espace des nombres irrationnels at à l'espace triadique de Cantor.

Đ. Kurepa (Zagreb).

Mruvka, S. On complete proximity spaces. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 587–590. (Russian)

Let X be a proximity space in the sense of Efremovič and Smirnov [Smirnov, Mat. Sb. N.S. **31(73)** (1952), 543–574; MR **14**, 1107]. The proximity structure of X can be defined by a family $\{\rho_\xi\}_{\xi\in\Xi}$ of pseudo-metrics ρ_ξ: for $A, B\subset X$, A is close to B if and only if $\rho_\xi(A, B)=0$ for all $\xi\in\Xi$. Let $\tau=\tau(X)$ be the smallest cardinal number of such an index class Ξ. For $A\subset X$, let

$$\delta_\xi(A)=\sup\{\rho_\xi(x, y): x, y\in A\}.$$

A system $\mathfrak{R}$ of subsets of X is said to satisfy Riesz's condition if, for all $\xi_1, \xi_2, \cdots, \xi_k\in\Xi$ and $\varepsilon>0$, there is an $A\in\mathfrak{R}$ such that $\max[\delta_{\xi_i}(A)]<\varepsilon$. X is said to be complete if every family of closed subsets of X with the finite intersection property and satisfying Riesz's condition has non-void total intersection. A number of theorems about proximity spaces in general and complete proximity spaces are proved. The main theorem is the following. Let $\mathfrak{m}$ be an infinite cardinal number, and let X be a complete proximity space having cardinal number $2^{\mathfrak{m}}$, having an open basis of cardinal number $\leqq\mathfrak{m}$, and such that $\tau(X)\leqq\mathfrak{m}$. Then there exists a class F of subsets of X having cardinal number $2^{2^{\mathfrak{m}}}$ such that no set in F is a continuous image of any other set in F. *E. Hewitt.*

Isiwata, Takesi. Structures of a uniform space X and $C(X)$. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **5** (1956), 174–184.

Let C denote the set of all real-valued continuous functions on a completely regular space X. For $C_1\subset C$, let μC_1 be the smallest uniform structure on X for which every function in C_1 is uniformly continuous. The main content of the paper may be summarized as follows: 1. Every totally bounded structure compatible with the topology on X is μB for some family B of bounded functions. 2. Every structure compatible with the topology and smaller than μC is of the form μC_1.

{Result 1 is known [Myers, Proc. Amer. Math. Soc. **2** (1951), 153–158; MR **12**, 727]. The following counter-example to 2 was pointed out to the reviewer by J. R. Isbell. Let X be the product of the closed interval [0, 1] by an infinite discrete space N, with all points $(0, n)$, $n\in N$, identified. The metric given by $d((s, m), (t, n))=s+t$ if $m\neq n$, and $=|s-t|$ if $m=n$, is not totally bounded. The metric uniform structure on X is smaller than μC, but it is not of the form μC_1 for any $C_1\subset C$, since every uniformly continuous function on X is bounded.} *M. Jerison.*

Koehler, F. A note on neighboring Jordan curves. Amer. Math. Monthly **64** (1957), 184–185.

The author uses a simple geometric technique to prove the following theorem. Let C and C' be Jordan curves satisfying the following conditions: (i) The origin is inside C'; (ii) the circle $|z|=b>0$ is inside C; (iii) if the points z_1, z_2 are on C, then one of the arcs of C connecting z_1 and z_2 lies within a circle of diameter $d=c|z_2-z_1|$, where c is a fixed constant greater than unity. Then it follows that if C' lies within an ε-neighborhood of C, for $\varepsilon<b/2c$, then C lies within an ε_1-neighborhood of C', with $\varepsilon_1>(2c+1)\varepsilon$ and $\varepsilon_1=O(\varepsilon)$ as $\varepsilon\to 0$. *M. Reade.*

McAuley, Louis F. On decomposition of continua into aposyndetic continua. Trans. Amer. Math. Soc. **81** (1956), 74–91.

Let $M(p)$ denote the set of all points x of a continuum M such that there does not exist an uncountable non-separated collection $M(p, x)$ of subsets of M such that each element of $M(p, x)$ separates p from x in M. Let H be the collection of point sets $M(p)$ for the various points p of M. After generalizing the notion of an upper semi-continuous collection of sets so that it will apply in a topological space which may fail to satisfy the first countability axiom, the author proves the theorem: The collection H is an upper semi-continuous collection of disjoint closed point sets filling up M; and furthermore, the hyperspace (R, H) whose points are elements of H is a connected, aposyndetic, and separable Hausdorff space. For $x\in g\in H$, let $S(g, x)$ denote the set of all points y of g such that there does not exist a subcollection C of H such that (1) C^* is the sum of a finite number of continua and (2) C^* separates x from y in M. Denote by G the collection of all point sets $S(g, x)$ for various elements g in H and various points x of g. The author shows that G satisfies the decomposition theorem stated above for the collection H. In case M is a compact metric continuum the generalization of upper semi-continuous collection in M coincides with the usual definition and the elements of G are the components of the elements of H. Thus the author obtains the result: If M is a compact metric continuum and G is the collection of components of the subsets $M(p)$ for various points of M, then G is an upper semi-continuous collection of disjoint continua filling up M; and furthermore, with respect to its elements as points, G is a compact aposyndetic metric continuum.

W. T. Puckett (Los Angeles, Calif.).

See also: Pontrjagin, p. 152.

Algebraic Topology

★ **Steenrod, Norman E. The work and influence of Professor S. Lefschetz in algebraic topology.** Algebraic geometry and topology. A symposium in honor of S. Lefschetz, pp. 24–43. Princeton University Press, Princeton, N. J., 1957. $7.50.

Ansprache, gehalten zur Feier des 70. Geburtstages von S. Lefschetz. — Verzeichnis der Abschnitte: The fixed-point formula; Product spaces and intersections; Coincidences; The intersection ring; The relative homology groups; Duality in a relative manifold; The Alexander duality theorem; The fixed-point theorem for a complex; Pseudo-cycles; Singular homology theory; Locally connected spaces; The cohomology ring; Subsequent developments; Concluding remarks. *H. Hopf* (Zürich).

Smale, Stephen. A note on open maps. Proc. Amer. Math. Soc. **8** (1957), 391–393.

A map $p:X\to Y$ is said to have the covering homotopy property for a point up to homotopy if given $f:I\to Y$ (I the unit interval) and a point $q\in p^{-1}f(0)$, there exists a map $\bar{f}:I\to X$ with $\bar{f}(0)=q$, $p\bar{f}(1)=f(1)$ and with $p\bar{f}$ homotopic to f under a homotopy leaving the end-points of I fixed. The purpose of the paper is to give a set of conditions on (X, p, Y) which implies the aforesaid property. These are as follows: the space X is locally arcwise connected and Hausdorff; Y is semilocally 1-connected and metric; the map p is open, proper, and onto. Several corollaries follow from the main theorem. *E. E. Floyd.*

Curtis, M. L.; and Fort, M. K., Jr. Homotopy groups of one-dimensional spaces. Proc. Amer. Math. Soc. **8** (1957), 577–579.

The authors prove that, if S is a one-dimensional

separable metric space, then $\pi_k(S)=0$ for $k>1$. This is a generalization of the classical result, where S is a one-dimensional complex. Actually the following stronger theorem is obtained: Let X be a locally connected continuum whose one-dimensional integral singular homology group is a torsion group. Let S be a one-dimensional separable metric space. Then any map $f:X\to S$ is null-homotopic. *J.-P. Meyer* (Baltimore, Md.).

Borsuk, K. On a concept of dependence for continuous mappings. Fund. Math. 43 (1956), 95–113.

The author, in this paper, introduces two notions, that of a function being dependent on a set of functions, and that of a finite set of functions being separable. Afterwards he proceeds to study these concepts.

Let Y be an ANR and X a compactum. Denote by Y^X the space of maps f: $X\to Y$ topologized in the usual fashion. A function $g\in Y^X$ is said to be dependent on the set $\Phi\subset Y^X$ provided that for every compact space $X'\supset X$ the existence of an extension over X' for every element of Φ implies the existence of an extension of g over X'. Letting $c(\Phi)$ denote the set of functions dependent on Φ one has immediately the following relations:

$$\Phi\subset c(\Phi)=cc(\Phi),$$
$$c(\Phi_1\cup\Phi_2)\supset c(\Phi_1)\cup c(\Phi_2),$$
$$c(\Phi_1\cap\Phi_2)\subset c(\Phi_1)\cap c(\Phi_2).$$

If in the preceding the set Φ contains a single function f, the elements of $c(\Phi)$ are called multiples of f.

The function $g\in Y^X$ is said to be dependent on the set $\Phi\subset Y^X$ in dimension m if for every compact space $X'\supset X$ such that the dimension of $X'-X$ is less than or equal to m, the existence of an extension over X' for every function $f\in\Phi$ implies the existence of an extension of g over X'. The set of functions dependent on Φ in dimension m will be denoted by $c_m(\Phi)$. The sets $c_m(\Phi)$ satisfy relations similar to those satisfied by the sets $c(\Phi)$, and in addition

$$c_m(\Phi)\supset c_{m+1}(\Phi)\supset c(\Phi).$$

Let $f_1, \cdots, f_k$ be a finite sequence of elements of Y^X. These functions are said to be separate if there exists a point $y\in Y$ and disjoint open subsets $G_1, \cdots, G_k$ of X such that $f_i(x)=y$ for every $x\in X-G_i$. The functions $g_1, \cdots, g_k$ belonging to Y^X are called separable if there exist separate functions $f_1, \cdots, f_k$ homotopic respectively to $g_1, \cdots, g_k$. Theorem: If X is a compact space of dimension less than $2n$, and Y is the n-sphere, then every finite sequence of functions $f_1, \cdots, f_k$ belonging to Y^X is separable.

Suppose $f_1, \cdots, f_k$ is a separable system of functions belonging to Y^X. Let $g_1, \cdots, g_k$ be a separate system of functions homotopic respectively to $f_1, \cdots, f_k$. Let $y\in Y$, and $G_1, \cdots, G_k$ be such that $g_i(x)=y$ for $x\in X-G_i$, where $G_1, \cdots, G_k$ is a sequence of disjoint open subsets of X. Define

$$g(x)=g_i(x),\ x\in G_i;\ g(x)=y,\ x\in X-\textstyle\bigcup_{i=1}^k G_i.$$

The function g is called a join of the functions $f_1, \cdots, f_k$. Theorem: If X is a compact space of dimension less than $2n-1$, and Y is the n-sphere, then for every system $f_1, \cdots, f_k\in Y^X$ the joins of $f_1, \cdots, f_k$ constitute a single component of Y^X. Theorem: If $f_1, \cdots, f_k\in Y^X$, dimension $X<2n$, Y is the n-sphere, and m is an integer such that dimension $X<m<2n$, then $f\in c(\{f_1, \cdots, f_k\})$ if and only if f is a join of certain multiples $g_1, \cdots, g_k$ of $f_1, \cdots, f_k$.

In addition to the preceding the author proves several other theorems and states some unsolved problems. *J. C. Moore* (Princeton, N.J.).

Toda, Hirosi. Reduced join and Whitehead product. J. Inst. Polytech. Osaka City Univ. Ser. A. 8 (1957), 15–30.

Let $\alpha\in\pi_{p+1}(S^{m+1})$ and $\beta\in\pi_{q+1}(S^{n+1})$. Barratt and Hilton [Proc. London Math. Soc. (3) 3 (1953), 430–445; MR 15, 643] have proved the following formula:

$$E^{n+1}\alpha\circ E^{p+1}\beta=(-1)^{(p-m)(q-n)}E^{m+1}\beta\circ E^{q+1}\alpha,$$

where E^r denotes r-fold iterated suspension. It follows that the element

$$\gamma=E^n\alpha\circ E^p\beta-(-1)^{(p-m)(q-n)}E^m\beta\circ E^q\alpha$$

is in the kernel of the suspension homomorphism. In the present paper, the author proves that

$$\gamma=\pm[\iota_{m+n+1}, \iota_{m+n+1}]\circ E^{2n}H\alpha\circ E^pH\beta,$$

where ι_{m+n+1} is a generator of $\pi_{m+n+1}(S^{m+n+1})$ and H denotes the generalized Hopf homomorphism of G. W. Whitehead [Ann. of Math. (2) 51 (1950), 192–237; MR 12, 847]. The author intends to use this formula to prove the non-existence of elements of $\pi_{31}(S^{16})$ of Hopf invariant one. *W. S. Massey* (Providence, R.I.).

Gugenheim, V. K. A. M. On supercomplexes. Trans. Amer. Math. Soc. 85 (1957), 35–51.

The author adapts Kan's abstract semi-cubical theory [Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 1092–1096; 42 (1956), 255–258; MR 18, 142] and develops the corresponding semi-simplicial scheme, which seems preferable in some respects.

In order to develop homotopy theory, it becomes necessary to study maps of type $\Delta^n\times\Delta^1\to A$ (or, more generally, $\Delta^n\times I^r\to A$), where A is a semi-simplicial complex. A convenient tool for such studies is the notion of a super-complex, which is the abstraction of the set of all semi-simplicial maps of type $\Delta^{n_1}\times\cdots\times\Delta^{n_k}\to A$ for all ordered sets of integers $(n_1, \cdots, n_k)$, together with certain face and degeneracy operators. Super-maps, the Kan condition, and fibre maps are introduced, and the analogues of Kan's theorems are proved in this context. The technique is also useful in the study of function-spaces; a sample theorem is: let X, Y be semi-simplicial complexes, A a subcomplex of X, and let Y satisfy the Kan condition. Then the map $p:Y^X\to Y^A$ given by $pf=f|A\times\Delta^q$, where f: $X\times\Delta^q\to Y$, is a fibre map. *J.-P. Meyer* (Baltimore, Md.).

★Smith, P. A. Generators and relations in a complex. Algebraic geometry and topology. A symposium in honor of S. Lefschetz, pp. 307–329. Princeton University Press, Princeton, N. J., 1957. \$7.50.

The purpose of the author is to attach to any connected simplicial complex a sequence $\cdots\to F_n\to\cdots\to F_1\to 1$ of groups and homomorphisms, of order two (image$\subset$kernel), such that the associated "homology groups" p_n (kernel modulo image, the image assumed normal) at least have some of the properties of the homotopy groups, e.g. the Hurewicz property h^n, namely that p_n is isomorphic to the nth homology group provided $p_1, \cdots, p_{n-1}$ are trivial. Two different constructions, I and II, are specified. They begin with F_1 as the group of simplicial loops in the complex, and proceed inductively, by constructions motivated by geometrical considerations. There are

associated maps into the homology groups, and operations of the paths in the complex (local system). Construction I satisfies all h^n, but gives trivial groups for $n>2$. Construction II has h^1, h^2, h^3 and is non-trivial in dimension 3; p_3(2-sphere) is infinite. The author notes that D. M. Kan has recently found a procedure to construct the homotopy groups for any simplicial complex.

H. Samelson (Ann Arbor, Mich.).

Gugenheim, V. K. A. M.; and Moore, J. C. Acyclic models and fibre spaces. Trans. Amer. Math. Soc. **85** (1957), 265–306.

The main object of this paper is to calculate the term E^2 in the singular homology spectral sequence of a fibre space by use of the theory of acyclic models of Eilenberg and MacLane [Amer. J. Math. **65** (1953), 189–199; MR **14**, 670]. This had been done previously by J. P. Serre [Ann. of Math. (2) **54** (1951), 425–505; MR **13**, 574], using cubical singular homology with degenerate singular cubes defined using the last coordinate only. The present paper shows that a filtration can be defined on the singular chains of a fibre space and the term E^2 of the resulting spectral sequence is as calculated by Serre in case one uses simplicial singular homology theory or a cubical singular homology theory with degeneracies defined on all coordinates.

A by-product of the authors' work is a proof that simplicial singular homology and cubical singular homology coincide even with local coefficients. For ordinary coefficients this had been proved by Eilenberg and MacLane [op. cit.].

In order to obtain these results, the authors add to the notion of a "category C with acyclic models" that of "degeneracy": C has a system of degeneracies if every map $f:M\to A$ in C, where M is a model, can be factored into the composition of maps $\alpha(f):M\to M'$ and $\beta(f):M'\to A$, where M' is a model and $\alpha(f)$ and $\beta(f)$ are required to satisfy certain axioms. For example, in Serre's cubical singular homology theory, with $M=$ an n-cube I^n, $\alpha(f)$ is the projection of I^n onto one of its faces and $\beta(f)$ is non-degenerate.

W. S. Massey (Providence, R.I.).

Kervaire, Michel. Courbure intégrale généralisée et homotopie. Math. Ann. **131** (1956), 219–252.

Es werden zunächst Abbildungen einer kompakten, differenzierbaren, orientierten Mannigfaltigkeit X_{n+k} in die Sphäre S_n betrachtet. Jede solche Abbildung läßt sich (Pontrjagin, Thom) beliebig gut durch Abbildungen f approximieren, welche im Urbild einer Umgebung eines gegebenen Punktes $c \in S_n$ differenzierbar mit maximalem Funktionalrang sind. Das Urbild von c ist dann eine (nicht notwendigerweise zusammenhängende) orientierte reguläre Mannigfaltigkeit M_k, welche ein stetiges Feld F_n von n unabhängigen Normalen trägt; umgekehrt gehört jedes solche Paar (M_k, F_n) zu einer Abbildung f der beschriebenen Art. Es wird nun, bei fester X_{n+k} im Bereich der Paare (M_k, F_n) ein Äquivalenzbegriff so eingeführt, daß äquivalenten (M_k, F_n) homotope f entsprechen (diese Charakterisierung der Homotopieklassen geht auf die alte Bemerkung von Pontrjagin zurück, daß bei hinreichender Regularität von f die Homotopieklasse von f bereits durch das Urbild einer infinitesimalen Umgebung eines Punktes bestimmt ist). Unter der Voraussetzung, daß entweder $k+1<n$ oder daß X_{n+k} eine Sphäre ist (bei beliebigen k, n), werden die Äquivalenzklassen der (M_k, F_n) zu einer Abelschen Gruppe gemacht, von der gezeigt wird, daß sie mit der Cohomotopiegruppe von Borsuk-Spanier zusammenfällt; für $X_{n+k}=S_{n+k}$ ist sie also die Homotopiegruppe $\pi_{n+k}(S_n)$. Nun wird angenommen, daß X_{n+k} differenzierbar im euklidischen Raum R_m liegt und ein Feld von $m-(n+k)$ unabhängigen Normalen trägt; dann bewirken für jedes Paar (M_k, F_n) die n Vektoren von F_n zusammen mit den längs M_k gegebenen $m-(n+k)$ Normalen von X_{n+k} eine Abbildung φ von M_k in die Stiefelsche Mannigfaltigkeit $V_{m,m-k}$ der orthogonalen $(m-k)$-Beine in einem Punkte des R_m. Da die $k$te Homologie- (=Homotopie-)Gruppe von $V_{m,m-k}$ bei geradem k unendlich zyklisch, bei ungeradem k zyklisch von der Ordnung 2 ist, ist die Homotopieklasse von φ durch eine ganze Zahl bezw. Restklasse mod 2, $\bar{\varphi}$, charakterisiert. Ferner sei χ^* die "Semicharakteristik" von M_k, d.h. bei geradem k die Hälfte der Euler-Poincaréschen Charakteristik, bei ungeradem k die mod 2 reduzierte Hälfte der Summe der Bettischen Zahlen. Es gilt der Satz: Die Differenz $\bar{\varphi}-\chi^*=I(M_k, F_n)=\gamma(f)$ ist eine Homotopieinvariante der Abbildung $f:X_{n+k}\to S_n$, und in den Fällen, in denen die vorhin genannten Cohomotopiegruppen existieren, ist sie ein Charakter dieser Gruppen. Weiter: Wenn $X_{n+k}=S_{n+k}$, also γ ein Charakter von $\pi_{n+k}(S_n)$ ist, so ist γ stabil bei Einhängung von f; bei geradem k ist immer $\gamma=0$. Der Verfasser spricht die Vermutung aus, daß im Falle $X_{n+k}=S_{2n-1}$ γ die mod 2 reduzierte sogenannte Hopfsche Invariante ist (und in einer nachträglichen Anmerkung teilt er mit, daß er diese Vermutung bewiesen habe). — Nunmehr kommt man zur Untersuchung der verallgemeinerten "Curvatura Integra", die, wie der Titel der Arbeit zeigt, ein Hauptziel des Verfassers ist. M_k sei topologisch, differenzierbar und mit n unabhängigen Normalenfeldern in den euklidischen R_{k+n} eingebettet; dann ist wie oben eine Abbildung $M_k\to V_{k+n,n}$ bestimmt, deren Homotopieklasse durch eine ganze Zahl bezw. Restklass mod 2, $\bar{\varphi}$, charakterisiert ist; $\bar{\varphi}$ ist die verallgemeinerte Curvatura Integra (für $n=1$ der Grad der Gauss'schen sphärischen Abbildung). Dann folgt aus oben angeführten Sätzen (wobei man R_{n+k} durch eine $S_{n+k}\subset R_{n+k+1}$ ersetzt): Bei geradem k ist $\bar{\varphi}=\chi^*$ [in Verallgemeinerung eines Satzes des Referenten, Math. Ann. **95** (1925), 340–367]. Dabei ist allerdings vorläufig vorausgesetzt (im Gegensatz zu dem alten Satz), daß M_k keine Selbstdurchdringungen hat; jedoch wird weiter gezeigt, daß man sich von dieser Voraussetzung befreien kann. Bei ungeradem k erhält man dieselbe Formel $\bar{\varphi}=\chi^*$ (mod 2) nur für M_k ohne Selbstdurchdringungen (wie in dem alten Fall $n=1$) und unter der weiteren Voraussetzung, daß $\gamma=0$ für jede Abbildung $S_{n+k}\to S_n$ ist. — Am Schluß der Arbeit werden noch einige Sätze über Tangentenfelder auf Mannigfaltigkeiten $M_k\subset R_{n+k}$ bewiesen, darunter der folgende: Jedes Cartesische Produkt von r Sphären, $r>1$, von denen wenigstens eine von ungerader Dimension ist, ist parallelisierbar.

H. Hopf (Zürich).

Sugawara, Masahiro. On a condition that a space is an H-space. Math. J. Okayama Univ. **6** (1957), 109–129.

The author proves in Theorem 2 that a space F is an H-space (homotopy-associative, with inversion) if it can be imbedded into a situation somewhat like the loop space of a space, as follows: Let $p:(E, F)\to(B, b)$ be a map that maps the relative homotopy groups isomorphically, and let E be contractible over itself with a stationary point ε; here E, F are a CW-complex and subcomplex, b is a point in B, ε is a vertex of F. In Theorem 1 the author gives a necessary and sufficient condition that F be an H-space, with a similar set-up; this time one assumes only F contractible in E, with ε stationary; the weak topology in

$F\times F$ is assumed to coincide with the product topology. This generalizes the results of E. H. Spanier and J. H. C. Whitehead [Comment. Math. Helv. **29** (1955), 1–8; MR **16**, 610]. In the course of the argument the author introduces three conditions, which are weakenings of the homotopy lifting condition for fibre spaces, and which ultimately are proved equivalent to each other. A tool used is a natural map from the join of an H-space F with itself to the suspension of F. *H. Samelson.*

Izbicki, Herbert. Reguläre Graphen 3., 4. und 5. Grades mit vorgegebenen abstrakten Automorphismengruppen, Farbenzahlen und Zusammenhängen. Monatsh. Math. **61** (1957), 42–50.

The author generalizes Frucht's construction [Canad. J. Math. **1** (1949), 365–378; MR **11**, 377] for a cubic graph with a given automorphism group. He shows that for any finite group G and any integers n, χ, c satisfying $3\leqq n\leqq 5$, $2\leqq\chi\leqq n$, $1\leqq c\leqq n$, $(\chi, c)\neq(2, 1)$, there exist infinitely many finite non-isomorphic graphs which are regular of degree n, have chromatic number χ and connectivity c, and which have automorphism groups isomorphic with G.
W. T. Tutte (Toronto, Ont.).

Dirac, G. A. A theorem of R. L. Brooks and a conjecture of H. Hadwiger. Proc. London Math. Soc. (3) **7** (1957), 161–195.

Brooks' theorem states that a k-chromatic graph having no vertex of degree $>k$ must have a subgraph which is a complete k-graph. Hadwiger's conjecture asserts that every k-chromatic graph can be reduced to a complete k-graph by omitting some edges and contracting others to single points. In this paper the author goes beyond Brooks' result by classifying the critical k-chromatic graphs having just one vertex of degree $>k$. He deduces that $2E\geqq(k-1)N+k-3$ for any critical k-chromatic graph, other than a complete k-graph, of E edges and N vertices. He finds also that Hadwiger's conjecture holds for any k-chromatic graph ($k\geqq5$) in which the vertices of degree $>k-1$ can be divided into two classes, one containing at most one vertex and the other containing no two vertices joined by an edge.

In the last part of the paper the author defines the excess of a k-chromatic graph as $2E-(k-1)N$. He then studies the number $e(k, n)$ defined as the greatest integer e such that every critical k-chromatic graph having no complete $(k-n)$-graph as a subgraph has an excess of at least e ($k\geqq3$; $0\leqq n\leqq k-3$). He proves that if $k\geqq4$ then $e(k, n)\geqq k-3+n$. *W. T. Tutte* (Toronto, Ont.).

Dirac, G. A. Short proof of a map-colour theorem. Canad. J. Math. **9** (1957), 225.

The author has proved that for $h>3$ and $h\geqq5$ a map on a surface of connectivity h with chromatic number $n_h=[\frac{1}{2}(7+\sqrt{(24h-23)})]$ always contains n_h mutually adjacent countries [same J. **4** (1952), 480–490; MR **14**, 394]. More recently he has shown that a critical k-chromatic graph of $N>k$ vertices and E edges satisfies $2E\geqq(k-1)N+k-3$ [see the paper reviewed above]. In the present note he gives a very simple derivation of the former result from the latter. *W. T. Tutte.*

See also: Dolbeault, p. 171; Denniston, p. 174.

GEOMETRY

Geometries, Euclidean and other

Sedláček, Jiří. On systems of diagonals in convex n-gons. Časopis Pěst. Mat. **81** (1956), 157–161. (Czech)

Let P be a convex n-gon in the plane no three diagonals of which have a common point in the interior of P. A system $S_k^{(n)}$ is a set of diagonals with exactly k points of intersection in P's interior and such that addition to $S_k^{(n)}$ of any further diagonal introduces at least one new point of intersection. Let $f(k, n)$ be the number of different systems $S_k^{(n)}$. It is shown that

$$f(0, n)=\binom{2n-4}{n-2}/(n-1),\quad f(1, n)=\tfrac{1}{2}(n-3)f(0, n),$$

and a more complicated formula is obtained for $f(2, n)$. The problem, for which k there exist systems $S_k^{(n)}$ and how $f(k, n)$ can be evaluated, remains unsolved for $k>2$.
F. A. Behrend (Melbourne).

Maravall Casesnoves, Darío. On the invariants common to a conic and the circumference of a circle. Gac. Mat., Madrid (1) **8** (1956), 199–207. (Spanish)

The author solves some elementary exercises on geometric loci. *P. Abellanas* (Madrid).

Frucht, Robert. Upper and lower bounds for the area of a triangle for whose sides two symmetric functions are known. Canad. J. Math. **9** (1957), 227–231.

If Δ is the area of a triangle of sides a, b, c, it is shown that

$$s(s-q)^2(s+2q)\geqq27\Delta^2\geqq s(s+q)^2(s-2q),$$

where

$$s=\tfrac{1}{2}(a+b+c),\quad q=+\sqrt{(a^2+b^2+c^2-bc-ca-ab)}.$$

This is shown to be a definite refinement at both ends on Beatty's result [Trans. Roy. Soc. Canada. Sect. III (3) **48** (1954), 1–5; MR **16**, 611], which in the present notation becomes

$$(s^2-q^2)^2\geqq27\Delta^2\geqq(s^2-q^2)(s^2-4q^2),$$

and to agree with the extremal values found by differential methods. *P. Du Val* (London).

ApSimon, H. G. Geodesic opposites on a regular tetrahedron. Math. Gaz. **41** (1957), 95–97.

On the surface of a regular tetrahedron, each point P has at least one "geodesic opposite" P' whose minimum geodesic distance from P is maximum. (It does not follow that P is a geodesic opposite of P'). The author finds the locus (on each face) of points Q whose geodesic opposites are not unique. The corresponding locus of Q' encloses regions within which Q' cannot lie. *H. S. M. Coxeter.*

Cavallaro, Vincenzo G. Triangles Brosteineriens. Mathesis **66** (1957), 34–39.

A triangle T is said by the author to be a "brosteinerian" of a given triangle ABC if the angles of T are equal to 2ω, $2\omega_1$, $2\omega_2$, where ω, ω_1, ω_2, are the Brocard angle and the two Steiner angles of ABC [For a definition of the Steiner angles see Rouché et Comberousse, Traité de géométrie, vol. I, 7th ed., Gauthier-Villars, Paris, 1900, p. 479].

If O, O_9 are the circumcenter and the ninepoint center

of ABC, and U the midpoint of the segment determined by the Brocard points of ABC, the triangle having for sides $(OU-O_9U, OU, OU+O_9U)$ is a brosteinerian of ABC, and the area of this brosteinerian triangle is equal to $3(\alpha^2-\beta^2)(4\alpha^2-\beta^2)^{\frac{1}{2}}/4\beta$, α, β being the semiaxes of the Brocard ellipse of ABC.

If in ABC the ratio of the circumradius to the radius of the Brocard circle is equal to 4, then ABC admits as a brosteinerian the right triangle of sides 3, 4, 5.

N. A. Court (Norman, Okla.).

Blanchard, René. Orthopôles et transversales réciproques. Mathesis 66 (1957), 39–43.

The author envisages two parabolas inscribed in a triangle T, their foci being the ends of a circumdiameter of T, and considers the tangents at the vertices of the two curves, their axes, and the reciprocal (or isotomic) transversals of the latter two lines. As a result he obtains supplements to, and generalizations of, propositions by R. Goormaghtigh, V. Thébault, and others, which appeared in recent issues of Mathesis in the form of questions. Here is an example taken at random: If δ_1 is the axis of a parabola inscribed in a triangle T and having its focus on the circumcircle (O) of T, the line δ_1 meets (O) again in the point of contact of δ_1 with the parabola determined by δ_1 and the sides of T.

The paper concludes with the following proposition: If Δ_i is one of four given lines Δ and T_i the triangle formed by the remaining three given lines, the orthopole of the circumdiameter of T perpendicular to the line Δ_i and the trace of Δ_i on that diameter are two points symmetrical with respect to the Newtonian line of the quadrilateral formed by the four given lines Δ. *N. A. Court.*

Goormaghtigh, R. Sur les ellipses orthopolaires et sur les ellipses tritangentes à une hypocycloïde à trois rebroussements. Mathesis 66 (1957), 44–54.

The author shows that the tricuspidal hypocycloid of Steiner, or, more briefly, the deltoid (H) which is known as the envelope of the Simson lines of a triangle (T), is also the locus of the orthopole, for (T), of the perpendicular dropped from a variable point P of the circumcircle (O) of (T) upon the Simson line of P for (T). Several other propositions are proved which are just as interesting. Perhaps the most important proposition is the following: If to the projections of a variable point P of the circumcircle (O) of a triangle (T) upon the sides of (T) we attach the masses m_1, m_2, m_3, the locus of the center of gravity of these masses coincides with the orthopolar ellipse of the point Q' which is the symmetric, with respect to the center O of (O), of the point Q whose barycentric (or areal) coordinates for (T) are m_1, m_2, m_3.

As the author points out, this proposition provides a definition of the orthopolar ellipses associate with a triangle (T) which does not involve the notion of the orthopole. This new definition may thus help to derive a considerable number of properties of the ellipses tritangent to the deltoid (H). Such properties have been found by various means by a number of writers.

The author treats the subject analytically using complex coordinates. The bibliography is scrupulous and competent. *N. A. Court* (Norman, Okla.).

★**Lagrange, René. Produits d'inversions et métrique conforme.** Gauthier-Villars, Paris, 1957. x+332 pp. 4000 francs.

From the introduction: "L'Ouvrage que nous présentons ici est l'arrangement d'un certain nombre d'articles publiés depuis quelques années dans des revues diverses, avec quelques compléments de détails. Il comporte deux sections. La matière de la première section, concernant les produits d'inversion, se trouve dans quatre Mémoires publiés depuis 1950; Acta Math. **82** (1950), 1–70; Bull. Sci. Math. (2) **74** (1950), 79–112; **75** (1951), 47–64; Ann. Sci. Ecole Norm. Sup. (3) **69** (1952), 83–108 [MR **14**, 195, 196, 896]. Les éléments de la deuxième section ont paru dans deux articles des Ann. Sci. Ecole Norm. Sup. **59** (1942), 1–42; **62** (1945), 385–417 [MR **7**, 24; **8**, 336] et un article des Acta Math. **86** (1951), 259–295 [MR **14**, 399].

"Le sujet est loin d'être nouveau, et les produits d'inversions, par exemple, ont fait l'objet de multiples études; cependant les auteurs s'intéressaient à ces produits en tant que transformations du group anallagmatique, donc de transformations linéaires orthogonales des coordonnées polysphériques. On se place ici à un point de vue plus géométrique, en ce sens que la transformation anallagmatique produite par un produit d'inversions est étudiée en fonction des sphères d'inversion successives. Les résultats sont susceptibles d'une forme simple et intéressante, et ce problème se prête admirablement à l'emploi de l'algorithme des points, plans et sphères, classique en géométrie différentielle anallagmatique."

It the reviewer's opinion the last part of the quotation is open to disagreement. The use of polyspherical coordinates does not necessarily obscure the geometrical situation. But it would have made the use of matrix algebra more readily available. Computation would have been simpler and the close relationship between conformal geometry in n dimensions and $(n+2)$-dimensional pseudo-euclidean geometry would have been exhibited.

It would be artificial to define (ordinary) orthogonalities as products of symmetries and then base a discussion of euclidean geometry on the study of symmetries and their products. The corresponding generation of the conformal group by means of inversions does not seem to be a more natural approach to conformal geometry.

However these criticisms may miss the purpose of the present volume, which may not have been intended as an introduction. At any rate it contains interesting contributions to the approach chosen by the author.

P. Scherk (Saskatoon, Sask.).

Holz, W. K. B. Der Dreipunkt und seine primitiven Lösungen: Dreieck, Dreikreis, Umkreis. Math. Naturwiss. Unterricht 8 (1955/56), 108–112.

L'auteur appelle 'tripoint irrégulier' d'un plan la figure formée par trois lignes arbitraires de ce plan, joignant deux à deux les trois points A, B, C de celui-ci. Si ces lignes sont constituées par des arcs de cercle, le tripoint est 'régulier'. La somme des trois angles au centre, comptés positivement dans le sens des aiguilles d'une montre, et des trois angles intérieurs, comptés positivement dans le sens contraire, d'un tripoint régulier est toujours égale à π. Le problème principal consiste à trouver un tripoint régulier dont on connait les longueurs des côtés circulaires. Ce problème a une infinité de solutions. Chaque solution est une solution 'primitive' si: a) Les angles critiques ont une structure critique, c'est-à-dire, le changement de leurs signes ne modifie en rien la figure. b) Chaque côté du tripoint ne contient que deux des trois sommets. c) La solution est à une seule détermination. On démontre alors que le problème principal n'a que trois et seulement trois solutions primitives, à

savoir: 1) Le triangle ordinaire. 2) Le cercle circonscrit au triangle ordinaire. 3) Le tripoint appelé 'triarc', constitué par trois arcs de cercle, deux à deux tangents entre eux aux sommets, et par suite chacun coupant orthogonalement le cercle circonscrit. *F. Şemin.*

Bieberbach, Ludwig. Zur Euklidischen Geometrie der Kreisbogendreiecke. Math. Ann. **130** (1955), 46–86.

Dans un livre [Das ebene obere Dreieck, Hagen, 1944] et dans l'article analysé ci-dessus, Holz a traité de l'extension de la Géométrie d'Euclide aux triangles, appelées par lui des 'tripoints réguliers', dont les côtés sont constitués par des arcs de cercle, et a émis l'opinion qu'en approfondissant la question on aboutirait certainement à des résultats intéressants. C'est ce que L. Bieberbach s'est proposé de vérifier dans son beau travail. Voici en quelques mots et sans trop de détails les résultats essentiels obtenus par ce dernier.

Un tripoint régulier est appelé 'triarc' si les cercles qui constituent ses côtés sont deux à deux tangents aux sommets A, B, C. Le cercle passant par A, B, C et les côtés circulaires se coupent alors orthogonalement. Si les côtés circulaires sont à l'intérieur de ce cercle orthogonal, on a un 'triarc inscrit' et dans le cas contraire un 'triarc circonscrit'. Soient $'a$, $'b$, $'c$ les longueurs des côtés, $'r_a$, $'r_b$, $'r_c$ les rayons des côtés, $'\alpha$, $'\beta$, $'\gamma$ les angles au centre correspondant aux côtés, ces neuf quantités ayant des valeurs algébriques. Soient enfin $'A$, $'B$, $'C$ les centres des côtés et $r_\Delta>0$ le rayon du cercle orthogonal. Dans tous les cas, on a d'abord

$$'\alpha+'\beta+'\gamma=(2n+1)\pi \ (n \text{ entier}),$$

et ensuite on a également dans le cas général

$$'a='r_a{}'\alpha=r_\Delta{}'\alpha \operatorname{ctg}('\alpha/2), \quad a=2r_\Delta \cos('\alpha/2)$$

et des relations analogues en faisant des permutations circulaires.

La fonction $y=x \operatorname{ctg} x$ et sa fonction inverse $x=C(y)$ jouent ici le rôle représenté par cos et arc cos dans la géométrie du triangle ordinaire. Dans les quatres paragraphes qui concernent respectivement: 1) les triarcs inscrits, 2) les triarcs circonscrits, 3) d'autres types de triarcs, 4) les triarcs inverses, l'auteur démontre plusieurs théorèmes. Mais pour ne pas alourdir notre compte rendu et pour éviter les répétitions nous ne donnerons que les résultats relatifs au premier paragraphe, qui donnerons sans doute une idée nette sur la portée des autres paragraphes. Th. 1: Dans un triarc inscrit à angle droit, on a

$$C('a/'b)+C('c/'b)=\pi/2$$

(Théorème de Pythagore, B étant le sommet de l'angle droit du triangle ordinaire ABC); et dans le cas général d'un triarc inscrit à angle aigu ou obtus, on a selon le cas

$$C('a/2r_\Delta)\mp C('b/2r_\Delta)+C('c/2r_\Delta)=\pi/2.$$

Th. 2: Dans tout triarc inscrit orienté négativement, la somme des longueurs des deux côtés quelconques est supérieure ou égale à celle du troisième côté. Th. 3: Etant donné un triarc inscrit ABC orienté négativement, le plus grand côté du triarc correspond au plus grand angle du triangle ordinaire ABC. Th. 4 [6, 7]: La condition nécessaire et suffisante pour que $'a$, $'b$, $'c$ vérifiant $'a\leqq'b$, $'c\leqq'b$, $'a+'c>'b$ soient les côtés d'un triarc inscrit à angle droit [aigu, obtus] orienté négativement est que

$$C('a/'b)+C('c/'b)=\pi/2 \ [<\pi/2, >\pi/2].$$

Th. 5. Il existe un seul triarc inscrit à angle droit orienté négativement dont les côtés droits soient donnés. Etc. *F. Şemin* (Istanbul).

Jacobsthal, Ernst. Bemerkungen zu der Arbeit des Herrn Bieberbach über Kreisbogendreiecke. Math. Ann. **132** (1956), 145–147.

Dans l'article analysé ci-dessus, L. Bieberbach utilise la propriété pour l'expression

$$F(x)=\frac{x\cos x(\sin x-x\cos x)}{(\sin x\cos x-x)^2}$$

d'être monotone décroissante dans l'intervalle ouvert $0<x<\pi/2$. Mais pour le voir, il est obligé de subdiviser l'intervalle en question et de faire des calculs numériques, tandis que l'auteur propose ici une autre démonstration qui consiste à faire voir que

$$(t/\sin t)-1<2[(1/t)-\operatorname{ctg} t]^2 \quad (0<t<\pi).$$

F. Şemin (Istanbul).

Al-Dhahir, M. W. On the Pappus configuration. Rev. Mat. Hisp.-Amer. (4) **17** (1957), 18–21.

The author considers a cycle of three triangles each inscribed in the next [G. Hessenberg, Grundlagen der Geometrie, de Gruyter, Berlin-Leipzig, 1930, p. 69]. His chief result is that if one pair of the three triangles is a pair of Desargues triangles, then so are the other two pairs. Such a relationship is, in fact, a necessary and sufficient condition for this self-dual configuration to be self-polar [see § 5.7, Ex. 2 of Coxeter, The real projective plane, 2nd ed., Cambridge, 1955, p. 75; MR **16**, 1143]. *H. S. M. Coxeter* (Toronto, Ont.).

Parker, E. T. On collineations of symmetric designs. Proc. Amer. Math. Soc. **8** (1957), 350–351.

The author shows that every collineation of a v, k, λ configuration permutes the points and the lines so that there exists a 1-1 correspondence between cycles of points and cycles of lines with each pair of cycles of the same length. This implies that an arbitrary group of collineations of a v, k, λ configuration has equally many transitive sets on the points and on the lines. Also, if the incidence matrix A of a v, k, λ configuration satisfies $PA=AQ$, where P and Q are permutation matrices, then the configuration has an incidence matrix A' such that $PA'=A'P$. *H. J. Ryser* (Columbus, Ohio).

Bishara, S.; and Amin, A. Y. On the locus of a point on a cubic surface from which an inscribed apolar triangle is projective. Proc. Math. Phys. Soc. Egypt **5** (1955), no. 3, 23–32 (1957).

Let ABC be an apolar triangle, relative to a cubic surface F, which is inscribed in F, P any point of F, and $A'B'C'$ the projection from P on F of the triangle ABC. The author shows that a necessary and sufficient condition for $A'B'C'$ to be apolar relative to F is that P belong to a sextic surface G. Some descriptive properties relative to F and G are treated. *P. Abellanas.*

★ **Grosche, Günter. Projektive geometrie. II.** Mathematisch-Naturwissenschaftliche Bibliothek, 8. B. G. Teubner Verlagsgesellschaft, Leipzig, 1957. ii+196 pp. DM 9.10.

[For Bd. I see MR **18**, 816.] Contents. (A) Curves of second order and curves of second class: (1) polarity defined by a conic; (2) reducible conics; (3) the self-polar

triangle; (4) Steiner's projective generation of a conic; (5) theorems of Pascal and Brianchon; (6) exercises; (7) affine classification of the conics; (8) affine properties of the conics; (9) passage from projective geometry to equiform geometry; (10) equiform classification of the conics; (11) exercises; (12) plane projectivities that leave an irreducible C_2 invariant. (B) Pencils of conics: (13) complex projectivities; (14) the characteristic equation of a C_2-pencil; (15) pole-conics of a C_2-pencil; (16) exercises. (C) Projective geometry in n-dimensional spaces: (17) the n-dimensional vector space V_n; (18) union and intersection of linear subspaces of V_n; (19) exercises; (20) the n-dimensional projective space Π_n over a field; (21) the $PGL(n+1, k)$; (22) correlations in Π_n; (23) involutory correlations; (24) exercises. Solutions of the exercises.

As the contents show, the book is an elementary and in some aspects an incomplete introduction to projective geometry. The author starts with the analytical definition of the conics, proves analytically Steiner's theorem, and by means of it proves geometrically the theorems of Pascal and Brianchon. Complex plane projectivities are then treated for a subsequent study of the pencils of conics. The distinct types of non-degenerate pencils of conics are explained by a direct discussion of the characteristic equation. The principal properties of the conics of the poles of a straight line relative to the conics of a pencil of the first type are treated in § 15. Projective n-dimensional space is defined by means of $(n+1)$-dimensional vector space, and hence the author devotes §§ 17 and 18 to an algebraic definition of vector spaces and a study of their principal properties as well as those of their group of automorphisms. In § 22 coordinates for hyperplanes are introduced by means of the coefficients of their equations, and then dualities are analytically defined. A correlation is defined as a product of a duality by a projectivity (collineation). In § 23 polarities and null-systems are defined. There are twenty one exercises for which solutions are included at the end of the book.

{The principal criticisms are the following. The author devotes more than five pages (83–88) to prove that there is a unique plane collineation that induces a given projectivity on a conic and to obtain its equations. No mention is made of the projective invariance of the distinct types of pencils of conics, nor, in the study of the pole-conics, of quadratic correspondences. In § 21, the author is somewhat dogmatic in using Klein's definition of geometry.} *P. Abellanas* (Madrid).

★ **van Spiegel, Izak Willem. Geometry of aggregates.** Van Gorcum and Comp. N.V. — G. A. Hak and Dr. H. J. Prakke, Assen, Holland, 1957. x+100 pp.

Im Anschluss an die Arbeiten von Schur, Reye und Veronese behandelt Verf. mit analytischen und synthetischen Methoden in 32 Theoremen die allgemeine Theorie der linearen Mannigfaltigkeiten (Aggregate) beliebiger Dimension, d.h. der Unterräume des projektiven S_R und des dualen Raumes, die durch projektive Verwandtschaften erzeugten algebraischen Mannigfaltigkeiten, und birationale Korrespondenzen. Im Mittelpunkt der Untersuchungen steht eine Verallgemeinerung der Segre'schen Mannigfaltigkeit im S_R, $R=(m+1)(n+1)-1$, als geometrischer Ort $[m;n]_h$ der Punkte (z) mit den homogenen Koordinaten $(z_{00}, \cdots, z_{0n}; z_{10}, \cdots, z_{1n}; \cdots; z_{m0}, \cdots, z_{mn})$, für die die Matrix $[z_{ij}]$ aus $m+1$ Zeilen und $n+1$ Spalten den Rang $\leq h+1$ hat. Die Mannigfaltigkeit $[m;n]_h$ ist algebraisch und enthält $[m;n]_k$ für $k\leq h$; für $k=0$ hat man die Segre'sche Mannigfaltigkeit $[m;n]_0$. $[m;n]_h$ kann auch synthetisch erzeugt werden durch $n+1$ projektiv bezogene projektive Unterräume S_m des S_R, die sich in wechselseitig unabhängiger Lage befinden. Daraus ergibt sich, dass $[m;n]_h$ zwei Scharen $[m|n]_h$ und $[n|m]_h$ von linearen Räumen enthält, die durch die Vereinigungsräume zugeordneter Unterräume S_h von S_m gebildet werden. Analoge Überlegungen gelten für die dualen Gebilde und liefern die kontravariant erzeugte Mannigfaltigkeit $]m;n[_h$, die die Scharen $]m|n[_h$ und $]n|m[_h$ trägt. Die Punktmannigfaltigkeit, die gebildet wird aus den Trägern der Unterräume der Scharen $]m|n[_h$, sei $\overline{]m|n[}_h$; sie lässt sich auch darstellen als eine kovariant erzeugte Mannigfaltigkeit $[m;n]_{m-h-1}$. Eine kovariante Korrelation $p_{\mu\nu}\xi^\mu\eta^\nu=0$ $(\mu=0, \cdots, m; \nu=0, \cdots, n)$, wo (ξ) resp. (η) die Koordinaten einer Hyperebene im S_m resp. S_n sind, kann vermöge der $(m+1)\times(n+1)$-Matrix $[p_{ij}]$ durch einen Bildpunkt im S_R repräsentiert werden; der Rang der Matrix legt die Lage des Bildpunktes bezüglich $[m;n]_h$ fest für $h=0, \cdots, \min(m, n)$. Entsprechendes gilt für die duale Relation $\beta^{\mu\nu}x_\mu z_\nu=0$. Deutet man darin (z) als einen Punkt im Parameterraum S_m, so bestimmt die Relation vermöge $\xi^\nu=\beta^{\mu\nu}x_\mu$ ein sog. m-dimensionales kontravariantes Aggregat, dessen Charakter durch den Rang von $[\beta_{ij}]$ bestimmt wird. Ein lineares System von m-dimensionalen Aggregaten im S_n wird analog definiert durch eine trilineare Relation $\omega^{\mu\rho\nu}x_\mu y_\rho z_\nu=0$. Verf. studiert weiter die Mannigfaltigkeiten, die sich als Schnitt von $]m;n[_h$, $]m|n[_h$ und $\overline{]m|n[}_h$ mit einem S_r ergeben, und ihre gegenseitigen Beziehungen. Insbesondere ergibt sich ein Zusammenhang des durch $\omega^{\mu\nu\rho}x_\mu y_\nu z_\rho=0$ $(\mu=0, \cdots, m; \nu=0, \cdots, n; \rho=0, \cdots, r)$ definierten linearen Systems von ∞^n kontravarianten Aggregaten im S_r mit einer Mannigfaltigkeit $]n|m[_0\frown S_r$. Die trilineare Relation kann interpretiert werden als eine $]m|n[_0\frown S_r$ und als eine $]r|n[_0\frown S_m$, woraus insbesondere eine birationale Korrespondenz zwischen $]m|n[_0\frown S_r$ und $\overline{]r|n[}_0\frown S_m$ folgt. Das Studium der linearen Systeme von eindimensionalen kontravarianten Aggregaten im S_3 geht zurück auf E. A. Weiss [Punktreihengeometrie, Teubner, Leipzig-Berlin, 1939], von ihm "Ebenenreihen" genannt, aber die von ihm entwickelten Methoden sind zur Übertragung auf höhere Dimensionen ungeeignet. Abschliessend wendet Verf. seine allgemeine Theorie an zur Herleitung der Resultate von Reye für Systeme von 2-dimensionalen kontravarianten Aggregaten im S_3; genauer betrachtet werden die Systeme $]1|2[_0\frown S_3$, $]2|2[_0\frown S_3$, $]3|2[_0\frown S_3$ und $]4|2[_0\frown S_3$. [Man vergleiche auch T. G. Room, The geometry of determinantal loci, Cambridge, 1938.]

R. Moufang (Frankfurt am Main).

See also: Lense, p. 114; Hartmanis, p. 115; Bentley, p. 173.

Convex Domains, Integral Geometry

Woods, A. C. The anomaly of convex bodies. Proc. Cambridge Philos. Soc. **52** (1956), 406–423.

Let K be a closed convex set in E^n containing the origin as interior point, and Λ a lattice with determinant $d(\Lambda)$. For each i $(i=1, \cdots, n)$ we define $\mu_i(\Lambda)$ as the greatest lower bound of those positive μ for which μK contains i linearly independent lattice points, put

$$\Delta(K)=\inf_\Lambda \mu_1^{-n}(\Lambda)d(\Lambda)$$

and define the anomaly of K as

$$M(K)=\sup_{\Lambda}\mu_1(\Lambda)\cdots\mu_n(\Lambda)\Delta(K)/d(\Lambda).$$

It is known that for K with the origin as center $1\leq M(K)\leq 2^{(n-1)/2-1/n}$, but no K with $M(K)>1$ is known (star shaped K with $M(K)>1$ are known). It is proved here that $M(K)=1$ for $n=3$ if K has the origin as center, and $M(K)=1$ for $n=2$ and arbitrary K. *H. Busemann.*

Kosiński, A. A proof of an Auerbach-Banach-Mazur-Ulam theorem on convex bodies. Colloq. Math. **4** (1957), 216–218.

The theorem referred to has become known as the "sack of potatoes" theorem. Roughly, it states that if one defines a potato in n dimensions to be a convex body whose diameter does not exceed 1, then, given any volume V of potatoes, there exists a cubical container whose side depends only on V which will contain those potatoes. More precisely: If $\mathscr{V}$ is a collection of n-dimensional convex bodies V_i with diameters $\leq D$ such that $V=\sum |V_i|<\infty$, there exists an n-dimensional cube with side determined by D and V in which $\mathscr{V}$ can be embedded in such a manner that no two of its members intersect. Here $|V_i|$ is the n-dimensional volume of V_i. The author states that no proof has previously been published, although he indicates the existence of other unpublished proofs.

The proof consists of two parts. First, it is proved that a convex body V_i with diameter D in n dimensions can be enclosed in a rectangular parallelepiped with sides $\leq D$ and volume $\leq n!\,|V_i|$. This reduces the problem from one involving general convex bodies to one involving rectangular parallelepipeds. The author then proves that any collection of rectangular parallelepipeds with sides $\leq D$ and total volume $\leq V$ can be imbedded in a rectangular parallelepiped of which $n-1$ sides are of length $3D$ and the last of length $(V+D^n)/D^{n-1}$. The proof proceeds by induction. A shortest side of each parallelepiped is chosen and all are oriented in a common direction for a final stacking. The parallellepipeds are ordered so that this shortest side decreases. Looking at the $(n-1)$-dimensional parallelepipeds obtained by projecting orthogonally to the direction mentioned, these are blocked off in groups such that each group has a total $(n-1)$-dimensional volume lying between D^{n-1} and $2D^{n-1}$. The induction hypothesis allows one to arrange each group of $(n-1)$-dimensional parallelepipeds in an $(n-1)$-dimensional cube $3D$ on a side. The corresponding groups of n-dimensional parallelepipeds may be contained in n-dimensional parallelepipeds of which $(n-1)$ sides are $3D$ and the last is capable of estimate. It is then shown that when these are stacked the total height does not exceed $(V+D^n)/D^{n-1}$.
J. W. Green (Los Angeles, Calif.).

Leech, John. Equilibrium of sets of particles on a sphere. Math. Gaz. **41** (1957), 81–90.

The author considers a number of particles constrained to move freely on the surface of a sphere, each pair being repelled (or attracted) by a force depending only on the distance between them. He proves that the particles will be in equilibrium for every law of force if and only if they are situated at the poles, of one or more types, of the rotations in one of the finite rotation groups [see, e.g. Coxeter, Regular polytopes, Methuen, London, 1948, pp. 53–55; MR **10**, 261; or Weyl, Symmetry, Princeton, 1952, pp. 77–79, 149–154; MR **14**, 16]. He omits all discussion of stability of the equilibrium, because this depends on the particular law of force. For instance, eight particles at the vertices of a cube can increase the distance between nearest neighbors by moving to the vertices of a square antiprism [Schütte und van der Waerden, Math. Ann. **123** (1951), 96–124; MR **13**, 61].
H. S. M. Coxeter (Toronto, Ont.).

See also: Birch, p. 125; ApSimon, p. 161.

Differential Geometry

★ **Kruppa, Erwin. Analytische und konstruktive Differentialgeometrie.** Springer-Verlag, Wien, 1957. vii +191 pp. $9.30.

This is really two books on Differential Geometry in one. The first part is a compact treatment of the classical theory of curves and surfaces in Euclidean three-space. The second part deals with "Constructive Differential Geometry" in a form not available elsewhere.

The first part needs little comment, for it is largely an exposition of the usual topics. The notation is that of Gauss with a few modifications to include Euclidean vectors and Christoffel symbols. Tensors and differential forms are not discussed. This material is included in the first 90 pages.

The second part will be unfamiliar to most differential geometers. Here the subject is treated as a branch of Synthetic Geometry, and methods of Projective and Descriptive Geometries are used. Since these methods cannot handle limits and derivatives in a precise way, they cannot be used for rigorous foundations. Nevertheless, they provide a good, intuitive understanding of the basic ideas; and for those who think better with figures than with formulas they are perhaps the best entry into the subject. The treatment is basically intuitive, but at critical points falls back upon analysis to establish key theorems. The text is lavishly illustrated with complicated drawings.

The contents are as follows: Einleitung — Grundbegriffe der Vektorrechnung. A. Analytische Differentialgeometrie — I. Raumkurven; II. Längen, Winkel und Flächeninhalte auf krummen Flächen; flächentreue und konforme Abbildungen; III. Krümmung der Flächen; IV. Biegung von Flächen; V. Windschiefe Strahlflächen und Ergänzungen zur Kurventheorie; VI. Strahlkongruenzen; VII. Strahlkomplexe. B. Konstructive Differentialgeometrie — VIII. Konstructive Ergänzungen zur Theorie der Kurven und Torsen; IX. Konstructive Ergänzungen zur Flächentheorie; X. Konstructive Ergänzungen zur Theorie der windschiefen Strahlflächen; XI. Konstructive Differentialgeometrie besondere Flächen und Kurven; XII. Das konforme und das projective Bild der nichteuklidischen Geometrien auf den Flächen konstanter Gausscher Krümmung; XIII. Kinematische Differentialgeometrie. *C. B. Allendoerfer.*

Bakel'man, I. Ya. Differential geometry of smooth manifolds. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 145–146. (Russian)

The article describes results which are proved in detail by the author in Uspehi Mat. Nauk (N.S.) **11** (1956), no. 2(68), 67–124 [MR **18**, 230]. *H. Busemann.*

Wróbel, T. H. On a certain form of Gołąb's equations. Biul. Wojskowej Akad. Tech. **6** (1953), 43–50. (Polish)

Vrubel, T. G. On a certain form of Gołąb's equations. Biul. Wojskowej Akad. Tech. 6 (1953), 101–108. (Russian)

Russian version of the above paper.

Havel, Václav. On the projective conception of translation-surfaces. Časopis Pěst. Mat. 81 (1956), 331–336. (Czech. Russian and English summaries)

If $x_i=f_i(v), y_i=g_i(w)$ are two curves through the origin O of a euclidean space, then $z_i=f_i(v)+g_i(w)$ represents a "translation-surface". Observing that x_i, y_i, $\frac{1}{2}z_i$ and the point at infinity of the line through x_i, y_i are harmonic, a projective generalization is obtained replacing the hyperplane at infinity by a finite hyperplane. *F. A. Behrend.*

Nádeník, Zbyněk. Über die zu den Bertrandschen Kurven analogen Flächen. Schr. Forschungsinst. Math. 1 (1957), 156.

A B-surface in 3-dimensional euclidean space is a special Weingarten surface whose Gaussian curvature K and mean curvature H satisfy an equation of the form $k_1K+k_2H+1=0$, where k_1 and k_2 are constants such that $k_1>k_2^2$. Each B-surface is associated with another B-surface. On every B-surface there exist two mutually orthogonal congruences of curves (B-congruences) whose geodesic curvatures ${}^ik$, normal curvatures $1/{}^iR$, geodesic torsions $1/{}^iT$ $(i=1, 2)$ satisfy ${}^1k=p/{}^1R+q/{}^1T+r_1$, ${}^2k=-p/{}^2T+q/{}^2R+r_2$, where p, q, r_1, r_2 are constants with $r_1^2+r_2^2\neq0$. The lines of curvature of a non-developable B-surface S are B-congruences if and only if a developable B-surface may be associated with S. The proofs of these theorems and other properties of B-surfaces appear in Czechoslovak Math. J. 5(80) (1955), 194–219 [MR 17, 656]. *A. Fialkow* (Brooklyn, N.Y.).

Steenbeckeliers, Guy. Sur une propriété caractéristique des surfaces minima. Acad. Roy. Belg. Bull. Cl. Sci. (5) 43 (1957), 155–158.

Les surfaces minima jouissent de la propriété caractéristique suivante. Si, en coordonnés symétriques, nous considérons la représentation de ces surfaces sur leur image sphérique, les génératrices rectilignes de la sphère, images des lignes minima u et v de (S), ont la direction des tangentes aux courbes coordonnés de nom opposé v et u. *Résumé de l'auteur.*

Šulikovskiĭ, V. I. On a method of normalization of a tensor of a net on a surface. Uč. Zap. Kazan. Univ. 115 (1955), no. 14, 53–59. (Russian)

By the tensor of a net on a two-dimensional surface the author understands a tensor a_{ik} from the differential equation $a_{ik}du^idu^k=0$, taking into account that this tensor is defined by giving the net and the parametrization (u^1, u^2) to within a (functional) multiplier. The problem consists in finding a factor $e^{2\lambda}$ such that the form $e^{2\lambda}a_{ik}du^idu^k$ will have null (gaussian) curvature; this normalization the author calls euclidean, while the tensor $e^{2\lambda}a_{ik}$ is called the E-tensor of the net. Making use of previously known results, to a great extent his own [same Zap. 112 (1952), no. 10], the author reduces the problem in the metric case to a 2nd order partial differential equation. {Reviewer's remark. This was to be foreseen from a comparison with the theory of conformal mapping of a surface into the plane.} Knowing one solution permits us to find final equations of the net by quadratures. The results so obtained are extended to important classes of nets, like: the Čebišev, the Ricci geodesic, the doubly parallel (each of the two families consists of geodesically-parallel curves), the equiareal, the asymptotic, the net of lines of curvature. *Ya. S. Dubnov* (RŽMat 1957, no. 821).

Gheorghiev, Gh. Sur la décomposition d'un complexe de droites en congruences remarquables. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I (N.S.) 1 (1955), 53–68. (Romanian. Russian and French summaries)

The subject of the present note is to solve, in a systematic way, the problem of decomposing a complex of straight lines into congruences. The complexes considered are 1) the general complex, 2) the special complex, 3) the semi-special and 4) the cylindrical complex. For the decomposition the system of Pfaff equations which defines the complex is enlarged by an additional Pfaff equation and the resulting differential consequences are studied.

In the case in which the congruences are of a special type, as e.g. normal congruences, the position of the foci, limit points, etc. are determined with respect to the center C of the ray r of the complex and the points P_1 and P_2 on r which are symmetrically situated about C at a distance equal to the curvature of the complex on r.

The author outlines, finally, how the special and semi-special complexes can be studied with the help of fields of unit vectors. These cases have been excluded from consideration in Acad. R.P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. 6 (1955), 105–113 [MR 18, 413] where some of the results of the present paper are announced. *R. Blum* (Saskatoon, Sask.).

Kovancov, N. I. Application of the ideas of nonholonomic geometry to a line complex. Ukrain. Mat. Ž. 6, 270–281 (1954). (Russian)

The author obtains equations for lines of curvature of the first and second kind, formulas for calculation of the Gaussian, total and mean curvature, and the equation of the Dupin indicatrix of a non-holonomic surface of the complex. The complex is referred to the trihedral $\mathbf{A}\mathbf{I}_1\mathbf{I}_2\mathbf{I}_3$, where $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$ are unit vectors along the principal normal, the binormal of the complex, and the ray issuing from $\mathbf{A}$, respectively. An elementary transformation of the trihedral is determined by the system

$$d\mathbf{A}=\omega_iI_i,\quad d\mathbf{I}_k=\omega_{ki}I_i\quad (i, k=1, 2, 3).$$

The set of integral curves of the (not completely integrable) equation $\omega_3=0$ is called a non-holonomic central surface of the complex $\{\mathbf{I}_3\}$. The normal component along $\mathbf{I}_3$ of the curvature of a curve on a central surface is called by the author the normal curvature of the curve. Curves with extremal normal curvature are called lines of curvature of the first kind. The intersections of $\omega_3=0$ with developable surfaces are called lines of curvature of the second kind of the surface $\omega_3=0$. The total and mean curvatures of a non-holonomic surface are defined as the product and the sum of the extremal normal curvatures. The element of area of the surface is defined by the equality $dS_3=|\omega_1\mathbf{I}_1\times\omega_2\mathbf{I}_2|=\omega_1\omega_2$, with the analogous definition of the element of area dS_3' of the spherical representation $d\mathbf{I}_3=\omega_{31}\mathbf{I}_1+\omega_{32}\mathbf{I}_2$. The author defines the Gaussian curvature as the ratio of the elements of area $K_{k_3}=dS_3'/dS_3$. On the plane of the vectors $\mathbf{I}_1$, $\mathbf{I}_2$ the Dupin indicatrix of the surface $\omega_3=0$ is constructed in the usual way and its equation is derived in the local system of coordinates.

It is shown that in an analogous way it is possible to obtain all these results for the non-holonomic surfaces

$\omega_1=0$, $\omega_2=0$ of the complexes $\{I_1\}$ and $\{I_2\}$ and the corresponding formulas are given. The author finds the interesting relation $K_{h_1}+K_{h_2}=K_{h_3}$ among the Gaussian curvatures of the three surfaces.

In a final paragraph geometric properties are given for a number of special complexes. The bibliography does not mention the work of S. S. Byušgens, "Geometry of a vector field" [MR **8**, 90], in which Byušgens discusses lines of curvature and Gaussian, total and mean curvature of a field associated with the complex.

N. I. Alekseev (RŽMat 1955, no. 4663).

Klapka, J. Über Beziehungen einer Kurve auf einer Fläche im projektiven Raum S_3 zu den Komplexen ihrer kanonischen Geraden. Schr. Forschungsinst. Math. **1** (1957), 158–161.

Im projektiven S_3 erzeugen die kanonischen Geraden 1. bzw. 2. Gattung einer Fläche (x), die keine Koinzidenzfläche ist, einen Strahlenkomplex Ω^{+1} resp. Ω^{-1}. Sei $[x]$ eine auf (x) liegende Kurve; die kanonischen Geraden 1. Gattung in den Punkten von $[x]$ erzeugen eine in Ω^{+1} enthaltene Geradenkongruenz Φ^{+1}, die kanonischen Geraden 2. Gattung in den Tangentialebenen an (x) in den Punkten von $[x]$ erzeugen eine in Ω^{-1} enthaltene Kongruenz Φ^{-1}. Bei der Klein-Segre'schen Abbildung des Systems aller Geraden des S_3 auf die Hyperquadrik Q des S_5 entspricht Φ^{+1} resp. Φ^{-1} die Regelfläche $\bar{\Phi}^{+1}$ resp. $\bar{\Phi}^{-1}$ von Q. Die Kurve $[x]$ heisst in Bezug auf Ω^{ε} $(\varepsilon=+1, -1)$ allgemein oder speziell, je nachdem der Bompiani'sche Abwickelbarkeitsindex von $\bar{\Phi}^{\varepsilon}$ im S_5 maximal ist oder nicht. Sei Σ^{ε} das System der in Bezug auf Ω^{ε} speziellen Kurven von (x). Verf. stellt die Differentialgleichung dieses Kurvensystems in Determinantenform auf und diskutiert die Gleichung. Insbesondere folgt: Die Gleichung des Systems Σ^{+1} ist nur dann von niedrigerem Gerade als 2 bezüglich $(\omega_1 d\omega_2-\omega_2 d\omega_1)$, wenn die kanonische Ebene im Flächenpunkt die kanonische Kurve 1. Gattung oskuliert. Verf. benutzt die Bezeichnungen des Buches von Fubini und Čech, "Introduction à la géométrie projective différentielle des surfaces", Gauthier-Villars, Paris, 1931.

R. Moufang (Frankfurt am Main).

Grove, V. G. On closed convex surfaces. Proc. Amer. Math. Soc. **8** (1957), 777–786.

The following theorem is proved: Let S and $\bar{S}$ be two closed orientable convex surfaces of class C''' imbedded in an Euclidean space E_3 of three dimensions, and possessing no parabolic points. Let h be a differentiable homeomorphism of S onto $\bar{S}$ such that: a) $II=\overline{II}$, II and $\overline{II}$ being the second fundamental forms of S and $\bar{S}$ respectively; b) the Gaussian curvatures of S and $\bar{S}$ are equal at corresponding points; c) orientations are preserved. Then h is a rigid motion. Proof uses some integral formulas, a method which goes back to G. Herglotz for the proof of the uniqueness of Weyl's problem.

S. Chern (Chicago. Ill.).

Aleksandrov, A. D. Uniqueness theorems for surfaces in the large. I. Vestnik Leningrad. Univ. **11** (1956), no. 19, 5–17. (Russian)

The paper lists without complete proofs various uniqueness theorems which follow from a slight generalization of the principal result in one of the author's earlier papers [same Vestnik **9** (1954), no. 8, 3–17; MR **17**, 493]. In E^{n+1} let S be a hypersurface of class C'', $k_1\geqq k_2\geqq\cdots\geqq k_n$ its principal curvatures, $\mathfrak{n}$ its unit normal, $\mathfrak{x}$ the vector from the origin of a Cartesian coordinate system to the point $(x_1, \cdots, x_n)$ on S. We say that S is convex if $k_i>0$ $(i=1, \cdots, n)$. We consider functions $\phi(k_1, \cdots, k_n, \mathfrak{n}, \mathfrak{x})$ of class C' with (*) $\partial\phi/\partial k_i\cdot\partial\phi/\partial k_j>0$ $(i, j=1, \cdots, n)$. For convex S we require (*) only for $k_i>0$. The relation to elliptic partial differential equations comes through the fact that, because of (*), ϕ, when symmetric in $k_1, \cdots, k_n$, becomes elliptic after its arguments have been expressed in terms of $u(x_1, \cdots, x_n)$ and its derivatives, where $z=u(x_1, \cdots, x_n)$ represents S. It is an important generalization beyond previous results that this remains frequently true for non-symmetric ϕ.

The following are samples of the many results: For a closed surface S_0, star-shaped with respect to the origin, let ϕ be invariant under dilation of S_0 from O. If for another closed star-shaped surface S the function ϕ has the same value at points on the same ray from O, then S originates from S_0 by a dilation from O. Applying this to $\phi=k_1\cdots k_n r^n(\mathfrak{n}\mathfrak{e})^{-1}$ yields that a convex surface is determined up to dilation from O when the area of its spherical image is given as function of the cone projecting a variable domain on S. Also, various characterizations of the sphere by Blaschke, Grotemeyer, Süss follow.

If the surface S is the boundary of a bounded domain and $\phi(k_1, \cdots, k_n)=\text{const}$, then S is a sphere. This contains very many well-known results as special cases: $\phi=k_1+k_2$ and convex S (Liebmann), general S (H. Hopf and Voss); arbitrary $\phi(k_1, k_2)$ and analytic S (A. D. Alexandrov); $\phi(k_1, \cdots, k_n)$ or $\phi(k_1^{-1}, \cdots, k_n^{-1})$ elementary symmetric and S convex (Süss, Hsiung, Chuan-Chih); analytic arbitrary $\phi(k_1, \cdots, k_n)$ (Voss, but not all his results are obtainable from the theorem).

If two convex surfaces S', S'' have the same closed hemisphere as spherical image and the same supporting function along their boundaries and if

$$\phi(k_1', \cdots, k_n', \mathfrak{n})=\phi(k_1'', \cdots, k_n'', \mathfrak{n})$$

for points of S', S'' with the same normal $\mathfrak{n}$, then S'' originates from S' by a translation. This implies: If for a closed convex surface S the function $\phi(k_1, \cdots, k_n, \mathfrak{n})$ has the same values at points with normals which are symmetric with respect to a plane P, then S has a plane of symmetry parallel to P.

Under more general hypotheses on ϕ the following holds in E^3: Let S^0, S' be piecewise analytic surfaces, where S^0 is convex and closed and S' is simply connected and closed (self-intersection allowed). If k_1^0, k_2^0 are the principal curvatures of S^0 at the point with normal $\mathfrak{n}$, and $\partial\phi/\partial k_1\cdot\partial\phi/\partial k_2>0$ for all k_i with $k_1=lk_1^0$, $k_2=lk_2^0$ and if $\phi(k_1', k_2', \mathfrak{n})=\phi(k_1^0, k_2^0, \mathfrak{n})$ for points of S' and S^0 with the same normal $\mathfrak{n}$, then S' is congruent to S^0. When S^0 is a sphere then the condition for ϕ (supposing it independent of $\mathfrak{n}$) becomes $\partial\phi/\partial k_1\cdot\partial\phi/\partial k_2>0$ for $k_1=k_2$ and $\phi=\text{const}$ on S', and the assertion is that S' is a sphere. For symmetric ϕ this is due to H. Hopf.

Remarks on various generalizations conclude the paper.

H. Busemann (Los Angeles, Calif.).

Gallarati, Dionisio. Sulle superficie di S_5 i cui piani tangenti si appoggiano a piani assegnati. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 329–338.

Si dà la classificazione proiettiva delle superficie differenziabili appartenenti ad uno spazio proiettivo S_5, i cui piani tangenti si appoggiano a piani dati, assegnando le equazioni e le caratteristiche proiettive delle superficie trovate.

Le superficie certate, se α è uno dei piani dati, stanno nei casi (di dimensione quattro) che hanno α come vertice. I casi più interessanti sono i seguenti: I piani tangenti alle superficie F di S_5 si appoggiano (a) a 4 o (b) a 5 piani, aventi a due a due un sol punto comune uno, a tre a tre non passanti per uno stesso punto nè contenuti in uno stesso S_4. (a) Si ottengono due famiglie di superficie (generalmente trascendenti) aventi le equazioni parametriche ($x_0, \cdots, x_5$ coordinate omogenee in S_5; m, n costanti):

(I) $x_0:x_1:x_2:x_3:x_4:x_5=1:u^m:v^m:(u+1)^m:(v+1)^m:(u-v)^m$,

(II) $x_0:x_1:x_2:x_3:x_4:x_5=$

$$1:u:v:u^m:v^m:u^{(1-n)m/(m-n)}v^{(m-1)n/(m-n)}.$$

(b) Si ottiene una nuova caratterizzazione della superficie di Veronese. Si dimostra infatti che essa è l'unica superficie appartenente ad S_5, irriducibile, i cui piani tangenti incontrino 5 piani dati, aventi la configurazione sopra specificata. *F. Gherardelli* (Firenze).

See also: Kervaire, p. 160; Yanenko, p. 168; Jeger, p. 169.

Manifolds, Connections

Yanenko, N. N. On the theory of the class of a Riemann space. Trudy Sem. Vektor. Tenzor. Anal. **10** (1956), 139–191. (Russian)

Given an analytic Riemannian metric

$$ds^2=\sum_{i,j=1}^{m} g_{ij}dx^i dx^j$$

defined on a manifold U_m, the class of the metric is defined to be the least integer q such that the metric can be realized locally by an imbedding of U_m as a surface in Euclidean $(m+q)$-space, E_{m+q}. For example, if $q=1$ then locally U_m is isometric to a hypersurface of E_{m+1}; every metric is of class $\leqq m+1$. The problem is to find invariants of the metric which will determine its class, e.g., $q=0$ if and only if $R_{ijkl}=0$. This involves both differential equations, i.e., the equations of Gauss and Codazzi-Peterson, and algebraic conditions. The author defines a series of such invariants $T_1, T_2, T_3, \cdots$ which are non-negative integers. They are obtained from the numbers $R_p=\max_{S_p\subset S_m} \operatorname{rank} \{\lambda_1{}^i\lambda_2{}^j\Omega_{ij}\}$, where S_m is the tangent space to U_m at a point p, S_p is a subspace of S_m, $\lambda_1{}^i$, $\lambda_2{}^j$ are the components of any two vectors in S_p, and

$$\Omega_{ij}=\sum_{k,l=1}^{n} R_{ijkl}\omega^k\omega^l$$

are the curvature forms of the metric. T_p is the least integer T such that $R_{T+1}<(T+1)\cdot p$ and $R_T\geqq T\cdot p$. The following are his principal results: (1) If the metric is of class $\leqq q$ and $T_q\geqq 2$, then it is of class exactly q, and the rank r and type t of the metric [in the sense of Allendoerfer, Amer. J. Math. **61**, no. 3 (1939) 633–644; MR **1**, 28] coincide with T_1 and T_q respectively. (2) If $T_q\geqq 3$, then necessary and sufficient conditions that it have class q are (a) that the equations of Gauss $\Omega_{ij}=\sum_{s=1}^{q}\psi_i{}^s\wedge\psi_j{}^s$ be solvable (moreover, algebraic conditions are given for this); (b) $(d\psi_i{}^s-\omega_i{}^j\wedge\psi_j{}^s\wedge\psi_{i_1}{}^1\wedge\cdots\wedge\psi_{i_q}{}^q=0$, $s=1, \cdots, q$; $i, i_j=1, \cdots, m$. (3) If $T_q\geqq 4$ then a necessary and sufficient condition that q be the class of the metric is the solvability of the equations of Gauss. (This sharpens a theorem of Allendoerfer, loc. cit.) Finally some results are given for the special case of metrics of class 2.
W. M. Boothby (Evanston, Ill.).

Auslander, L.; and Kuranishi, M. On the holonomy group of locally Euclidean spaces. Ann. of Math. (2) **65** (1957), 411–415.

Let $\mathfrak{M}$ be the set of all compact n-dimensional riemannian manifolds with zero curvature and torsion tensor. The authors show that necessary and sufficient conditions for a finitely generated group P to be the fundamental group of an M in $\mathfrak{M}$ are: (1) P contains a free abelian normal subgroup on n generators, N, which is maximal abelian; (2) P/N is finite; and (3) P has no elements of finite order. The necessity of (1) and (2) are due to Bieberbach [Math. Ann. **70** (1911), 297–336; **72** (1912), 400–412]. In order to prove the sufficiency, the authors show that P is isomorphic to a subgroup of the rigid motions of euclidean n-space and that the orbit space is in $\mathfrak{M}$. Lastly, let G be any finite group and let $G=F/R$, with F and R free non-abelian. Set $F_0=F/[R, R]$, $R_0=R/[R, R]$. The authors then show that F_0, with $R_0=N$, fulfills (1)–(3): (1) is a recent result of M. Auslander and Lyndon [Amer. J. Math. **77** (1955), 929–931; MR **17**, 709] and (3) is demonstrated by group cohomology. [For a purely group-theoretical proof that F_0 satisfies (3) cf. G. Higman, Quart. J. Math. Oxford Ser. (2) **6** (1955), 250–254; MR **19**, 117]. Since P/N is the holonomy group of the manifold, it then follows that any finite group $G=F_0/R_0$ is the holonomy group of some M in $\mathfrak{M}$. *A. Rosenberg.*

Lelong-Ferrand, Jacqueline. Quelques propriétés des groupes de transformations infinitésimales d'une variété riemanienne. Bull. Soc. Math. Belg. **8** (1956), 15–30.

Let V be a finite Riemannian manifold with boundary δV. The author considers Lie derivatives X with respect to a vector field ξ, and shows that X considered as an operator on differential forms has an adjoint which is a Lie derivative if ξ is a Killing field, i.e., is associated with an infinitesimal isometry. She then shows that in this case a harmonic form is invariant with respect to X if it has a tangential (or normal) component on the boundary which is invariant. This result is used to discuss the infinitesimal conformal transformations of a Kählerian manifold.

The author also establishes the inequality

$$\|X\phi\|\leqq B[\|d\phi\|+\|\delta\phi\|]$$

for an infinitesimal isometry X. *H. L. Royden.*

★ **Nomizu, Katsumi. Remarques sur les groupes d'holonomie et d'isotropie.** Colloque de topologie de Strasbourg, 1954–1955, Institut de Mathématique, Université de Strasbourg. 4 pp.

It is well known that the restricted homogeneous holonomy group of a symmetric Riemannian space is contained in the linear isotropy group at every point [E. Cartan, Bull. Soc. Math. France **54** (1926), 214–264]. The object of the present paper is to establish that conversely the above-stated condition is sufficient for the Riemannian space in consideration to be symmetric. In order to prove this theorem, the author proves first the theorem: If a homogeneous space G/H of a connected Lie group G (H: structure group) admits an invariant affine connection of which the homogeneous holonomy group ψ is contained in the linear isotropy group $\tilde{H}$, then the covariant derivatives of the torsion and curvature tensors are equal to zero. *A. Kawaguchi* (Sapporo).

Mutō, Yosio. On conformally curved Riemann spaces V_n, $n \geq 6$, admitting a group of motions G_r of order $r > n(n+1)/2-(3n-11)$. J. Math. Soc. Japan 9 (1957), 38–61.

For Riemannian manifolds V^n ($n \geq 6$) whose group of motions has dimension $> \frac{1}{2}n(n+1)-3n+11$, the author finds conditions that the Weyl conformal curvature tensor (assumed different from zero) has to satisfy. The method is to write down the equations expressing the invariance of the conformal curvature tensor $C_{\alpha\beta\gamma\delta}$ under an infinitesimal motion ξ^k at a point, and to note that they can be considered as linear equations for the values of the covariant derivatives $\xi^k{}_{;\lambda}$ at the point. Assuming that there are fewer than $3n-11$ independent equations one draws conclusions about the algebraic form of the $C_{\alpha\beta\gamma\delta}$. The conditions given are a set of linear equations for the $C_{\alpha\beta\gamma\delta}$ for $n=6, 8$, and a parametric representation, with a pair of orthogonal unit vectors as independent variables, of all possible $C_{\alpha\beta\gamma\delta}$ for the other n-values. In particular, the matrix C_{j11k}, $2 \leq k \leq n$, can be assumed, diagonalized, and it is shown that exactly $n-3$ of the eigenvalues must be equal to each other for $n > 8$, with slight modifications of this for 6, 7, 8. *H. Samelson.*

Egorov, I. P. Riemann spaces of second lacunarity. Dokl. Akad. Nauk SSSR (N.S.) 111 (1956), 276–279. (Russian)

Debever, R. Sur les espaces de Riemann à quatre dimensions à courbure totalement dégénérée. Acad. Roy. Belg. Bull. Cl. Sci. (5) 42 (1956), 1033–1044.

This paper follows others by the author [same Bull. (5) 42 (1956), 313–327, 608–621; MR 18, 761] in which he discussed the curvature invariants arising from the curvature tensor R_{ijkl} of a Riemannian space. The space is said to have degenerate curvature if the quadratic $R_{ijkl}p^{ij}p^{kl}$ in the bivector components p^{ij} has less than maximum rank, and the space has totally degenerate curvature if the quadratic has rank 1, i.e., if $R_{ijkl}=a_{ij}a_{kl}$ for some simple bivector a_{ij}. Canonical forms for the metrics of 4-spaces of totally degenerate curvature are given and are related to results given by H. S. Ruse and the reviewer for spaces of recurrent curvature.

A. G. Walker (Liverpool).

Géhéniau, Jules. Une classification des espaces einsteiniens. C. R. Acad. Sci. Paris 244 (1957), 723–724.

In an Einstein space of four dimensions we have $R_{pq}=0$, and the Riemann tensor R_{pqrs} is equal to the Weyl tensor C_{pqrs}. With the correlation

$$(23, 31, 12, 14, 24, 34)=(1, 2, 3, 4, 5, 6),$$

either may be exhibited as a 6×6 matrix

$$C=\tfrac{1}{2}(C^+ + C^-), \quad C^{\pm}=\begin{pmatrix} A^{\pm} & \pm A^{\pm} \\ \pm A^{\pm} & A^{\pm} \end{pmatrix},$$

where A^+ is a 3×3 symmetric complex matrix with vanishing trace, and A^- is its complex conjugate; local Minkowskian coordinates are used, so that $g_{pq}=\delta_{pq}$. The author states that it suffices to consider C^+, and he suppresses the sign $(+)$. He states that by a rotation of the orthonormal frame, A undergoes a transformation of the group o of rotations of complex three-dimensional Euclidean space, and hence the classification of spaces is reduced to the classification of pairs of conics $X_\alpha X_\alpha=0$, $A_{\alpha\beta}X_\alpha X_\beta=0$ with $A_{\alpha\alpha}=0$, $\alpha, \beta=1, 2, 3$. Spaces are divided into categories according to the characteristics of the elementary divisors of $A-aI$, where I is the unit matrix, and each category is divided into classes according to the roots a_1, a_2, $a_3=-a_1-a_2$ of the equation $|A-aI|=0$. The various cases are considered in detail. Reference is made to J. Géhéniau and R. Debever, Acad. Roy. Belg. Bull. Cl. Sci. (5) 42 (1956), 114–123 [MR 17, 1016]; J. Géhéniau, ibid. 42 (1956), 252–255 [MR 17, 1144]; R. Debever [see the paper reviewed above and references therein]. {See also A. Z. Petrov [Kazan. Gos. Univ. Uč. Zap. 114 (1954), no. 8, 55–69; MR 17, 892]. The exposition in the paper under review is very brief, and it is not clear to the reviewer why $A_{\alpha\beta}{}^+X_\alpha X_\beta$ is invariant, as appears to be assumed; it seems rather that only this quantity plus its complex conjugate is invariant.}

J. L. Synge (Dublin).

Sen, R. N. Note on non-simple K^*-spaces. Proc. Nat. Inst. Sci. India. Part. A. 22 (1956), 82–85.

K^*-spaces have been studied by Walker [Proc. London Math. Soc. (2) 52 (1950), 36–64; MR 12, 283] and Ruse [Publ. Math. Debrecen 2 (1952), 169–174; MR 14, 1124; for a review cf. Schouten, Ricci calculus, 2nd ed., Springer, Berlin, 1954, p. 421; MR 16, 521]. In this paper the defining equations of a non-simple K^*-space are replaced by equations containing the sum of the curvature tensors of two suitably chosen spaces which are conformal to the K^*-space. The equations for the case where the K^*-space is recurrent have been obtained first, and these have been modified later (under certain assumption) for the case where the K^*-space is either recurrent or symmetric.

J. A. Schouten (Epe).

Jeger, M. Ueber Inflexionen in projektiven Zusammenhaengen. Univ. e Politec. Torino. Rend. Sem. Mat. 15 (1955–56), 201–224.

Bekanntlich gibt es im Fall von differenzierbaren Abbildungen zwischen zwei projektiven Ebenen α und $\bar{\alpha}$ in jedem Punkt P im allgemeinen drei ausgezeichnete Richtungen mit der Eigenschaft, dass eine Inflexion in P dann und nur dann erhalten bleibt, wenn die Kurventangente in einer von diesen Richtungen verläuft. In der vorliegenden Arbeit wird dieser Satz und einige weitere verwandte Erkenntnisse durch den Beizug von projektiven Zusammenhängen in einem gewissen Sinne verallgemeinert. In der Ebene α sei ein projektiver Zusammenhang etwa durch die projektiven Parameter Π^i_{jk} beschreiben. Ein Punkt $P(x^1, x^2)$ einer Kurve c, in dem der Ausdruck

$$\ddot{x}^{[1}\dot{x}^{2]}-\Pi^{[1}_{jk}\dot{x}^{2]}\dot{x}^j\dot{x}^k$$

verschwindet, wird als Inflexionspunkt in bezug auf den projektiven Zusammenhang erklärt. Die erwähnte Verallgemeinerung des obigen Satzes besteht darin, dass es zu jedem Paar von projektiven Zusammenhängen in jedem Punkt P im allgemeinen drei ausgezeichnete Richtungen gibt, mit der Eigenschaft, dass eine Kurve dann und nur dann eine Inflexion in bezug auf beide projektive Zusammenhänge aufweist, wenn die Kurventangente in einer von diesen Richtungen verläuft. An diesen Satz knüpfen weitere interessante Resultate des Verf. an, insbes. über Eigenschaften der Geodätischen von projektiven Zusammenhängen. *O. Borůvka* (Brno).

Pisareva, N. M. On the fractional quadratic integral of geodesic lines in n-dimensional spaces of affine connection. Dokl. Akad. Nauk SSSR (N.S.) 108 (1956), 198–200. (Russian)

Mocanu, P. Espaces partiellement projectifs. Rev. Math. Pures Appl. 1 (1956), no. 1, 67–98.

A translation from the Romanian of the article reviewed in MR 18, 67.

Complex Manifolds

★ **Behnke, Heinrich. Funktionentheorie auf komplexen Mannigfaltigkeiten.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 45–57. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

This expository paper, written by one of the leading experts in the field, gives a concise but lucid account of the great advances accomplished in recent decades in the theory of analytic functions of n complex variables. In the case $n=1$, the concept of a Riemann surface played a central role throughout the development of the theory. After a compact discussion of modern developments in the case $n=1$, the author proceeds to guide the reader through the elusive issues relating to the case $n>1$ where there is no perfect substitute for the concept of a Riemann surface. The paper contains a careful account of the fundamental concepts and results due to F. Hirzebruch, K. Stein, H. Cartan, L. Calabi, B. Eckmann, J. P. Serre, M. Rosenlicht, K. Oka, P. Thullen, E. E. Levi, F. Norguet, H. Bremermann, W. Thimm, W. L. Chow, K. Kodaira, E. Kreyszig, H. Grauert, R. Remmert, W. Stoll, and the author himself for the case $n>1$. *T. Radó.*

Remmert, Reinhold. Projektionen analytischer Mengen. Math. Ann. 130 (1956), 410–441.

Le mémoire poursuit l'étude des sous-variétés analytiques complexes (que nous appellerons avec l'auteur ensembles analytiques) et donne des conditions pour qu'un ensemble analytique soit projeté sur un sous-espace selon un ensemble analytique, en utilisant les résultats d'un travail précédent [Remmert und Stein, Math. Ann. 126 (1953), 263–306; MR 15, 615]. Un ensemble analytique M est dit uniformisable en un point $z_0 \in M$ si chaque composante irréductible de M en z_0 est uniformisable en z_0; cette définition ne suppose pas M irréductible en z_0 et permet d'énoncer: si M est de dimension complexe homogène d, il existe un voisinage de z_0, $U(z_0)$, et un ensemble analytique M^*, de dimension au plus $d-2$, qui contient les points non uniformisables de $M \cap U(z_0)$. Une fonction f continue sur M y est dite holomorphe si elle l'est en chaque point uniformisable, et sur chaque composante en ce point, comme fonction des paramètres d'uniformisation locaux; si f est holomorphe aux points de M qui n'appartiennent pas à un ensemble analytique M', où $\dim M' \leq d-1$, f est holomorphe sur M. Si M est irréductible et de dimension d en O, $C^{n-d}(z_{d+1}, \cdots, z_n) \cap M$ ayant O comme point isolé, une fonction $W(z)$ holomorphe sur M en O est algébroïde et solution d'équation $w^s + \sum_0^{s-1} c_i w^i = 0$, les fonctions $c_1(z_1, \cdots, z_d)$ étant holomorphes à l'origine et le pseudo-polynome étant irréductible. La restriction d'une fonction holomorphe f sur M à un ensemble analytique $M' \subset M$ est holomorphe sur M'. Le résultat n'est pas trivial, M' pouvant ne contenir que des points où M est non uniformisable, et est obtenu à partir de l'énoncé précédent, et de théorèmes de R. Remmert et K. Stein (cf. mémoire cité). Un sous-ensemble $N \subset M$ est analytique sur M en $z_0 \in M$, s'il est défini en annulant des fonctions holomorphes sur M au voisinage de z_0. Si M est un ensemble analytique dans un domaine G de C^n, si $N \subset M$ est un ensemble analytique sur M, N est aussi un ensemble analytique dans G. Une représentation $z=\varphi(x)$, $x \in M$, $z \in M'$, d'un ensemble analytique $M \in C^n$ dans $M' \in C^n$ est dite holomorphe si les $z_\nu = \varphi_\nu(x_i)$ sont holomorphes sur M.

Une structure complexe sur un espace de voisinages X, ou espace analytique complexe, est alors déterminée essentiellement par les données suivantes: (a) un recouvrement de X par des cartes (U_j, ψ_j) d'un ensemble (U), $j \in J$, ensemble d'indices; $U_j \subset X$ est ouvert et représenté biunivoquement par ψ_j dans C^n sur un ensemble analytique. (b) U_i, U_j sont comparables, c'est-à-dire: $\psi_j \psi_i^{-1}$ est holomorphe sur $U_i \cap U_j$. (c) Si (U, ψ) est une carte comparable avec tous les (U_i, ψ_i), elle appartient à (U). Les notions de point uniformisable et de fonction holomorphe sur X sont définies à partir de la représentation locale ψ_i. Une représentation τ d'un espace complexe X sur un espace complexe Y est dite holomorphe si $f \circ \tau$ est holomorphe pour toute fonction f holomorphe sur un ouvert $V \subset Y$. Un ensemble analytique A dans X peut alors être considéré comme un sous-espace complexe de X (on dira que A est plongé analytiquement dans X) par restriction à A de la structure complexe de X. Si $\check{X}$ est l'ensemble des points uniformisables de X, supposé de dimension homogène, l'ensemble $B(m) \subset \check{X}$ où le rang d'un système $(f_1, \cdots, f_s)$ de fonctions holomorphes sur X vaut au plus m, a pour fermeture $\bar{B}(m)$ dans X un ensemble analytique dans X; la démonstration utilise l'extension suivante d'un énoncé de Remmert et Stein: si N est un ensemble analytique dans X, si $\dim N \leq l < d$, si M est un ensemble analytique de dim. homogène d dans $X-N$, la fermeture $\bar{M}$ de M dans X est un ensemble analytique de dim. homogène d dans X. On considère alors un ensemble analytique A dans le produit $X \times Y = Z$ de deux espaces analytiques complexes et on étudie la projection $q(A)$ de A sur Y selon $(x \times y) \to y$. Si $y=q(z)$, la fibre $F(y)=q^{-1}[q(z)]$ est pour $z \in A$ un ensemble analytique dans Z et dans X. Si A est de dim. homogène s dans Z, si les fibres $F(y)$ relatives à A sont de dim. homogènes d, à chaque point $(x, y) \in A$ correspondent des voisinages U_x, V_y tels que la projection de $A \cap (U_x \times V_y)$ sur V_y soit un ensemble analytique de dim. homogène $s-d$ dans V_y. On en déduit une condition suffisante pour que $q(A)$ soit localement un ensemble analytique dans Y. On appelle rang au point $z \in A$ de la projection $q(A)$ le nombre $s(z)-d(z)$, différence des dim. en z de A et de la fibre $q^{-1}[q(z)]$. Si A est de dimension homogène, l'ensemble $A(m)$ des points $z \in A$ en lesquels le rang de la projection q ne dépasse pas m est un ensemble analytique dans $X \times Y$. On obtient sous la forme suivante l'énoncé principal du mémoire: si A est un ensemble analytique dans $X \times Y$, si à chaque point $y \in q(A)$ de la projection on peut associer un voisinage $W(y)$ et un ensemble X_y relativement compact dans X, de manière que chaque composante de $F(\eta)$ pour $\eta \in q(A) \cap W(y)$ rencontre X_y, alors la projection $q(A)$ de A sur Y est localement un ensemble analytique dans Y, et est de dim. homogène r, r étant le rang de la projection q; $q(A)$ est alors un ensemble analytique dans Y chaque fois qu'il est compact, en particulier si A est lui-même compact, ou si l'on a $A = A \cap (X' \times Y)$, X' étant compact sur X, et plus particulièrement si X est compact lui-même; si Y est un espace projectif, $q(A)$ est alors algébrique, cas qui correspond à un problème classique d'élimination algébrique.

P. Lelong (Paris).

Remmert, Reinhold. Meromorphe Funktionen in kompakten komplexen Räumen. Math. Ann. **132** (1956), 277–288.

Dans ce travail, on donne des démonstrations simples des deux théorèmes suivants: Théorème 1 [Thimm, Math. Z. **60** (1954), 435–457; MR **16**, 582]: soit X un espace analytique (complexe) compact et irréductible; si des fonctions méromorphes dans X sont analytiquement dépendantes, elles sont algébriquement dépendantes; — Théorème 2 (Chow, annoncé sans démonstration): si X est un espace analytique (complexe), compact et irréductible, de dimension n, le corps $K(X)$ des fonctions méromorphes dans X est une extension algébrique simple d'un corps de fractions rationnelles à k variables, avec $k \leqq n$.

Principe de la démonstration: soient k fonctions méromorphes f_i $(1 \leqq i \leqq k)$ dans X, analytiquement indépendantes dans le cas du th. 1, resp. formant une base de transcendance du corps $K(X)$ dans le cas du th. 2 (si on admet le th. 1, on a $k \leqq n$ dans l'hypothèse du th. 2). Dans les deux cas, il existe un espace analytique compact et irréductible Y, et une application analytique propre π de Y sur X, telle que l'application $f \to f \circ \pi$ soit un isomorphisme de $K(X)$ sur $K(Y)$, et que les $f_i \circ \pi$ n'aient pas de point d'indétermination dans Y; on peut donc se ramener au cas où les f_i n'ont aucun point d'indétermination. Alors les f_i définissent une application analytique $F: X \to \bar{C}^k$, en notant $\bar{C}$ la sphère de Riemann. D'après un théorème de Remmert [voir l'article analysé ci-dessus] l'image de F est un sous-ensemble analytique (fermé) de $\bar{C}^k$, donc un sous-ensemble algébrique d'après un théorème classique de Chow. Si les f_i sont analytiquement dépendantes, le "rang" de l'application F est $<k$; $F(X)$ est alors de dimension $<k$, donc l'ensemble algébrique $F(X)$ est $\neq \bar{C}^k$, d'où le théorème de Thimm. On montre ensuite (par un raisonnement de pure topologie) que si $m(z)$ désigne le nombre des composantes connexes de la fibre $F^{-1}(z)$ du point $z \in \bar{C}^k$, il existe un ouvert non vide $B \subset \bar{C}^k$ et un entier m, tels que $m(z) \leqq m$ pour $z \in B$ (l'auteur affirme même, sans démonstration, qu'on peut choisir m de manière que $m(z) \leqq m$ pour $z \in \bar{C}^k$); si f est méromorphe dans X, f dépend algébriquement de $f_1, \cdots, f_k$ par hypothèse, et on voit alors que f est racine d'un polynôme de degré $\leqq m$, ce qui démontre le th. 2. On notera que le degré du corps $K(X)$ sur le corps engendré par $f_1, \cdots, f_k$ est au plus égal au nombre maximum des composantes connexes des fibres de l'application F définie par $f_1, \cdots, f_k$. *H. Cartan* (Paris).

Dolbeault, Pierre. Formes différentielles et cohomologie sur une variété analytique complexe. II. Ann. of Math. (2) **65** (1957), 282–330.

Pour la première partie, voir MR **18**, 670]. Soit V une variété analytique complexe de dimension complexe m.

Chapitre III. (Sur les formes méromorphes fermées.) A) Une p-forme méromorphe est de seconde espèce si elle est localement cobord d'une $(p-1)$-forme méromorphe. Si $p>1$, toute p-forme méromorphe est de seconde espèce, si son ensemble polaire est une variété (sans point singulier). Si $p=1$, on peut définir le résidu d'une 1-forme méromorphe; c'est un diviseur $W = 2i\pi \sum A_k W_k$, A_k constantes complexes, W_k composantes irréductibles de l'ensemble polaire de la forme; la forme est de seconde espèce si et seulement si son résidu est nul (même si l'ensemble polaire a des singularités). Tout germe de diviseur est résidu d'un germe de 1-forme méromorphe fermée, défini modulo les germes de 1-formes holomorphes fermées. B) Si W est un diviseur, il définit alors un élément de la 1-cohomologie à valeurs dans le faisceau des germes de 1-formes holomorphes fermées, donc un élément de $K^{1,1}(V, C)$ (chapitre I; quotient de l'espace des courants fermés de degré total 2, de $\bar{z}$-degré $\leqq 1$, par le sous-espace des cobords des courants de degré total 1), donc un élément de $H^2(V, C)$. Par ailleurs W définit un 1-1-courant fermé [de Rham and Kodaira, Harmonic integrals, Inst. Advanced Study, Princeton, 1950; MR **12**, 279], donc un élément de $K^{1,1}(V, C)$; c'est le même que celui qui précède. Enfin à W on peut associer un cycle singulier w (chap. II), donc une classe d'homologie de dimension $2m-2$, donc par dualité un élément de $H^2(V, C)$; c'est encore celui qui est défini ci-dessus. Pour que W soit résidu d'une 1-forme méromorphe fermée, il faut et il suffit que l'élément correspondant de $K^{1,1}(V, C)$ soit nul. Ensuite l'auteur donne des conditions pour qu'une 2-forme holomorphe fermée soit cobord d'une 1-forme méromorphe fermée; par exemple, si V est de Stein, toute 2-forme holomorphe fermée est cobord d'une 1-forme méromorphe fermée, si le 2e nombre de Betti est fini.

Chapitre IV. (Formes méromorphes et courants associés). Le reviewer a montré [Schwartz, Géométrie différentielle, Colloq. Internat. Centre Nat. Rech. Sci., Strasbourg, 1953, pp. 185–195; MR **16**, 518] que, si ω est une forme différentielle semi-méromorphe (quotient d'une forme C^∞ par une fonction holomorphe) dont l'ensemble polaire Γ est une variété, on peut définir un courant associé vp ω (vp=valeur principale de Cauchy). Si D est un opérateur différentiel semi-holomorphe (par exemple $D=d_z=d'$), $D(\text{vp }\omega)=\text{vp }D\omega$; $d_{\bar{z}}\omega = d''\omega$ est un courant porté par Γ. Kodaira a montré [Amer. J. Math. **73** (1951), 813–875; MR **13**, 981] la même propriété si $m=2$, quand Γ a des singularités. Ici l'auteur étend la définition à m quelconque, pourvu que les points singuliers de Γ forment une variété Γ_2, et que quelques autres conditions de régularité soient vérifiées par Γ et Γ_2. La méthode est une combinaison de celles de Kodaira et du reviewer; les calculs sont difficiles. On a encore $D(\text{vp }\omega)=\text{vp }D\omega$; $d_{\bar{z}}$ vp ω est un courant porté par Γ, et qui lui-même fait intervenir la vp d'une forme semi-méromorphe définie sur Γ. Si ω est fermée de degré 1, $d_{\bar{z}}$ vp ω est le courant associé au résidu de ω (chap. III, B). L'auteur définit alors le résidu d'une p-forme méromorphe fermée, p quelconque, pourvu que son ensemble polaire Γ vérifie les conditions précédentes; si ω est de seconde espèce, son résidu est nul. L'auteur donne des conditions pourqu'un $(p, 1)$-courant soit un résidu, généralisant celles du chapitre III. *L. Schwartz* (Paris).

Halder, Gita; and Behari, Ram. Orthogonal ennuples in a Kaehler manifold. Proc. Nat. Inst. Sci. India. Part A. **22** (1956), 305–315 (1957).

In a local Kaehler structure over coordinates $\{z^\alpha, z^{\alpha^*}\}$ the authors envisage the formal modification of coordinates $\zeta^\alpha = \varepsilon i z^\alpha$, $\zeta^{\alpha^*} = -\varepsilon i \bar{z}^\alpha$, $\varepsilon = +1$ or -1, and study some of its properties. (i) If the modification carries geodesics into geodesics, then in the defining vector field the covariant coordinates are holomorphic. (ii) If to an n-tuple of orthogonal contravariant self-adjoint vector fields, any two of which are inequivalent under the modification, one adds such equivalents, then the total $2n$-tuple is a complete orthogonal system; and the $2n$-tuple forms a normal congruence of curves if and only if the covariant components of all the vectors are holomorphic. (iii) If such covariant components are holomorphic then all Ricci-coefficients γ_{rst} are 0, and if the assumption occurs only for one member of the n-tuple

and its equivalent, then a suitable subset of the γ_{rst} is 0.
S. Bochner (Princeton, N.J.).

★ **Andreotti, Aldo. On the complex structures of a class of simply-connected manifolds.** Algebraic geometry and topology. A symposium in honor of S. Lefschetz, pp. 53–77. Princeton University Press, Princeton, N. J., 1957. $7.50.

In the Cartesian product of the projective plane $P^{(2)}:(x_0, x_1, x_2)$ and the projective line $P^{(1)}:(y_1, y_2)$, Hirzebruch [Math. Ann. **124** (1951), 77–86; MR **13**, 574] considered the surfaces $\Sigma_n:x_1y_1^n-x_2y_2^n=0$ ($n=0, 1, 2, \cdots$), and proved that they are all analytically distinct, although Σ_m and Σ_n are differentiably homeomorphic whenever $m\equiv n$ (mod 2). In the present paper the author complements the work of Hirzebruch by proving that any irreducible, non-singular algebraic surface which has 3-genus $P_3\neq 25$ and which is differentiably homeomorphic to one of the surfaces Σ_n is actually birationally equivalent, without exceptions, to one of the surfaces Σ_n. He further proves that any irreducible non-singular algebraic surface which is differentiably homeomorphic to the projective plane $P^{(2)}$ is birationally equivalent, without exceptions, to $P^{(2)}$. The discussion is based on the interesting preliminary theorem that the surfaces Σ_n, considered as lying in the Segre variety which is the image of $P^{(2)}\times P^{(1)}$ in $P^{(5)}$, can be obtained by birational projection, without exceptions, from a ruled surface of degree $n+2$ with a directrix line, in $P^{(n+3)}$. The author includes several geometric applications of the above results as well. One of these, related to a theorem of M. Noether, is that every irreducible algebraic surface with a linear pencil of rational curves can be birationally transformed into a rational ruled surface so that the generators are the images of the curves of the pencil. Another application is the theorem that every non-singular rational surface without exceptional curves of the first kind is birationally equivalent, without exceptions, either to the projective plane or to a rational normal ruled surface of even order with a directrix line (that is to say, to one of the surfaces Σ_n with n even).
R. C. Gunning (Princeton, N.J.).

Atiyah, M. On the Krull-Schmidt theorem with application to sheaves. Bull. Soc. Math. France **84** (1956), 307–317.

The Krull-Schmidt theorem asserting the existence and essential uniqueness of direct sum decompositions into indecomposable factors is proved in exact categories in the sense of Buchsbaum [Trans. Amer. Math. Soc. **80** (1955), 1–34; MR **17**, 579] satisfying a suitable chain condition. As an application it is shown that the Krull-Schmidt theorem holds in the class of vector bundles over a connected complete algebraic variety or over a connected compact complex manifold. This is achieved by showing that the exact category whose objects are suitable sheaves satisfies the chain conditions.
S. Eilenberg.

Atiyah, M. F. Complex analytic connections in fibre bundles. Trans. Amer. Math. Soc. **85** (1957), 181–207.

Ziel des Verf. ist, die charakteristischen Klassen eines komplex-analytischen Prinzipal-Faserbündels als Hindernisse gegen die Existenz eines komplex-analytischen Zusammenhangs zu definieren.

Unter (P, X, G) werde immer ein komplex-analytisches Prinzipalbündel verstanden, wo die Basis X eine komplexe Mannigfaltigkeit und die Strukturgruppe G eine komplexe Liesche Gruppe ist. Auch Vektorraum-Bündel sollen immer komplex-analytisch sein. Es sei T das Bündel der kontravarianten Tangentialvektoren von X und Q das Vektorraum-Bündel über X, das als Punkte der Faser Q_x ($x \in X$), diejenigen über P_x definierten Felder von Tangentialvektoren von P hat, welche invariant sind unter den Rechtstranslation von G auf P. Schliesslich sei $L(P)$ das Vektorraum-Bündel über X, das die Liesche Algebra von G als Faser hat und zu P vermöge der adjungierten Darstellung von G assoziiert ist. Dann hat man die exakte Sequenz $A(P)$:

$$0\to L(P)\to Q\xrightarrow{g} T\to 0.$$

Einen komplex-analytischen Zusammenhang von P definiert Verf. als eine Zerfällung (splitting) dieser Sequenz, d.h. als einen komplex-analytischen Homomorphismus $h:T\to Q$, so dass $gh=1$: $T\to T$. Der Einfachheit halber werde in diesem Referat die q-te Cohomologiegruppe von X mit Koeffizienten in der Garbe der Keime von holomorphen Schnitten eines Vektorraum-Bündels W mit $H^q(X, W)$ bezeichnet. Ein komplex-analytischer Zusammenhang h ist ein Element von $H^0(X, \mathrm{Hom}(T, Q))$, welches unter g_* in das Element 1 von $H^0(X, \mathrm{Hom}(T, T))$ übergeht. Wendet man auf $A(P)$ den Funktor $\mathrm{Hom}(T, \ )$ an, dann erhält man eine exakte Sequenz von Vektorraum-Bündel und die zugehörige exakte Sequenz der Garben von Keimen von holomorphen Schnitten, zu der eine exakte Cohomologiesequenz gehört. Unter dem Corand-Operator dieses Sequenz geht 1 in ein Element $a(P)$ aus $H^1(X, \mathrm{Hom}(T, L(P)))$ über. Wenn man die vorstehenden Überlegungen im differenzierbaren Fall durchführt, dann ist $a(P)=0$, da die Garbe der Keime von differenzierbaren Schnitten eines Vektorraum-Bündels fein ist. Es existiert immer ein differenzierbarer Zusammenhang, der dann einen Krümmungstensor hat, durch den die charakteristischen Klassen berechnet werden können. Im komplex-analytischen Fall ist $a(P)$ im allgemeinen nicht null; ein komplex-analytischer Zusammenhang existiert dann und nur dann, wenn $a(P)$ verschwindet. Das Element $a(P)$ spielt in der ganzen Arbeit eine entscheidende Rolle.

Unter einem invarianten Polynom von $L(G)$ wird bekanntlich ein symmetrischer C-Homomorphismus von $L(G)\otimes_C L(G)\otimes_C\cdots\otimes_C L(G)$ in C verstanden, der unter jedem Element von $\mathrm{ad}(G)$ invariant ist. Für das folgende werde vorausgesetzt, dass $G=\mathrm{GL}(n, C)$ oder dass G halbeinfach ist. Es sei G_1 eine maximale kompakte Untergruppe von G. Die differenzierbaren G_1-Prinzipalbündel entsprechen eineindeutig den differenzierbaren G-Prinzipalbündel und die charakteristischen Klassen eines G-Prinzipalbündels P können einmal als die charakteristischen Klassen des entsprechenden G_1-Prinzipalbündels definiert werden, andererseits aber auch mit Hilfe der invarianten Polynome und des Krümmungstensors eines differenzierbaren Zusammenhangs von P. (Die charakteristischen Klassen sind dabei immer als Cohomologieklassen mit komplexen Koeffizienten aufzufassen.) $a(P)$ gehört zur Dolbeaultschen Gruppe vom Typ (1,1) mit Koeffizienten in $L(P)$ $[H^1(X, \mathrm{Hom}(T, L(P)))=H^{1,1}(X, L(P))]$. Wenn X eine kählersche Mannigfaltigkeit ist, dann liefern die invarianten Polynome angewandt auf $a(P)$ Elemente der Gruppen $H^{q,q}(X, C)$, welche mit Untergruppen von $H^{2q}(X, C)$ zu identifizieren sind. Hauptsatz des Verf. ist, dass man auf diese Weise genau die charakteristischen Klassen von P erhält. Als Folgerung ergibt sich, dass die charakteristischen Klassen (mit komplexen Koeffizienten) eines komplex-analytischen Prinzipalbündels P verschwinden, wenn P einen komplex-analytischen Zusammenhang

zulässt. Zum Beweis seines Hauptsatzes zeigt Verf., dass für einen differenzierbaren Zusammenhang vom Typ (1, 0) das Element $a(P)$ gleich der d''-Cohomologieklasse der (1, 1)-Komponente des Krümmungstensors ist.

Verf. untersucht besonders noch den Fall $G=\mathrm{GL}(n, C)$. Dann handelt es sich um die Chernschen Klassen. In diesem Fall kann man den Hauptsatz durch die Aufspaltungsmethode [Hirzebruch, Neue topologische Methoden in der algebraischen Geometrie, Springer, Berlin, 1956; MR 18, 509] auf Geradenbündel zurückführen. — Die Chernschen Klassen sind für $\mathrm{GL}(n, C)$-Prinzipalbündel oder für die assoziierten Vektorraum-Bündel definiert, welche den lokal-freien kohärenten komplex-analytischen Garben entsprechen. Verf. zeigt, wie man das Hindernis $a(P)$ auch im Falle beliebiger kohärenter komplex-analytischer Garben definieren kann.

Auf die interessanten Anwendungen kann Ref. nur kurz eingehen. Für ein Vektorraum-Bündel E über einer algebraischen Kurve X (assoziiert zum $\mathrm{GL}(n, C)$-Prinzipal bündel P) kann $a(P)$ durch den Serreschen Dualitätssatzes als Element $b(E)$ des zu $V=H^0(X, \mathrm{Hom}(E, E))$ dualen Vektorraumes V^* aufgefasst werden. Verf. zeigt, dass $b(E)$ auf den nilpotenten (globalen) Endomorphismen von E verschwindet und auf den Automorphismen von E den Wert $2\pi i\cdot\deg(E)$ annimmt, wo $\deg(E)$ den Grad der ersten Chernschen Klasse von E bezeichnet. Nun wird E dann und nur dann von einer Darstellung $\pi_1(G)\to\mathrm{GL}(n, C)$ induziert, wenn P einen komplex-analytischen Zusammenhang zulässt (der auf einer algebraischen Kurve automatisch einen verschwindenden Krümmungstensor hat, also integrierbar ist). Daraus ergibt sich der folgende Satz von A. Weil [J. Math. Pures Appl. (9) 17 (1938), 47–87]:

Es sei $E=E_1\oplus E_2\oplus\cdots\oplus E_q$ die Remaksche Zerlegung von E in nicht weiterzerlegbare Summanden ($E_i\neq 0$), die dann eindeutig bestimmt sind. E wird dann und nur dann von einer Darstellung der Fundamentalgruppe induziert, wenn $\deg(E_i)$ für $i=1, 2, \cdots, q$ verschwindet.

[Vgl. auch die oben referierten Arbeit.]

F. Hirzebruch (Bonn).

See also: Lelong-Ferrand, p. 168.

Algebraic Geometry

★**Hodge, W. V. D. Professor Lefschetz's contributions to algebraic geometry: an appreciation.** Algebraic geometry and topology. A symposium in honor of S. Lefschetz, pp. 3–23. Princeton University Press, Princeton, N. J., 1957. $7.50.

Ansprache, gehalten zur Feier des 70. Geburtstages von S. Lefschetz. — Verzeichnis der Abschnitte: Integrals of the second kind on an algebraic variety; The subvarieties of an algebraic variety; The theory of correspondences; Abelian varieties; Conclusion. *H. Hopf* (Zürich).

d'Orgeval, B. A propos du cône de Véronèse. Bull. Soc. Roy. Sci. Liège 25 (1956), 369–370.

The anticanonical system on the Veronese cone (threefold cone in S_6 projecting the Veronese surface from a point) consists of what the author calls Du Val surfaces, i.e. cubic sections of the cone residual to a quadric cone. It is proved that such a surface, all of whose genera are unity, is nevertheless not in general birationally equivalent to any quartic surface in S_3. *P. Du Val.*

Bentley, Beverley. The multeity locus in projective space of fifteen dimensions. J. London Math. Soc. 31 (1956), 471–478.

Room [Amer. J. Math. 74 (1952), 967–984; MR 14, 837] described a linear system of 7 collineations in S_{15} and a V_{10} of points which are fixed for some collineations of the system. The system also determines a symmetrical relation, an extension of the ordinary duality relation, between points, S_4's, S_{10}'s, and hyperplanes. The present paper is a more detailed study of V_{10}, which is shown to be of order 12 and birationally mapped on S_{10} by quadrics through a $G_6{}^5$ in a hyperplane, $G_6{}^5$ being the Grassmannian of lines in S_4. *P. Du Val* (London).

Godeaux, Lucien. Remarques sur la formation des systèmes canonique et pluricanoniques de quelques surfaces algébriques. Acad. Roy. Belg. Bull. Cl. Sci. (5) 43 (1957), 90–97.

Bruthans, Vladimír. Anallagmatic quintics. Časopis Pěst. Mat. 80 (1955), 274–283. (Czech)

The author determines all quintics which are invariant under an involutory quadratic transformation. Examples are given of quintics permitting two such transformations and hence a group of algebraic transformations.

F. A. Behrend (Melbourne).

Mahel, Vladimír. Über Kurven sechster Ordnung die in quadratischen Inversionen mit einem gemeinsamen Hauptpunktdreieck invariant sind. Časopis Pěst. Mat. 80 (1955), 284–298. (Czech. Russian and German summaries)

The investigations of the paper reviewed above are carried out for curves of order 6 in the case where the quadratic transformations have three fundamental points.

F. A. Behrend (Melbourne).

Rosati, Mario. Qualche aspetto della teoria delle funzioni ellittiche modulari ed abeliane modulari. Archimede 8 (1956), 145–153.

Klassifizierung der algebraischen Kurven mittels birationellen Invarianten. Für Kurven vom Geschlechte $p=1$ (elliptische Kurven) die birationell äquivalent mit einer ebenen Kurve dritter Ordnung ohne Doppelpunkt sind, kann man dazu erstens den algebraischen Modul $\mathfrak{J}$ benutzen, d.h. eine geeignete symmetrische Funktion der sechs verschiedenen Doppelverhältnisse von vier Tangenten aus einem Punkte der kubischen Kurve an diese Kurve. An zweiter Stelle ergibt sich eine tranzendente Invariante als der Quotient $z=\omega_1/\omega_2$ der primitiven Perioden des der ebenen Kurven zugehörigen elliptischen Integrals. Zwei elliptische Kurven sind birationell äquivalent wenn die zugehörigen Werte z und z' äquivalent in Bezug auf die modulare Substitution $z'=(\alpha z+\beta)/(\gamma z+\delta)$ ($\alpha, \beta, \gamma, \delta$ ganz; $\alpha\delta-\beta\gamma=1$) sind, und umgekehrt. Aus diesen Betrachtungen ergibt sich, dass der algebraische Modul $\mathfrak{J}$ der Beziehung $\mathfrak{J}(z)=\mathfrak{J}((\alpha z+\beta)/(\gamma z+\delta))$ genügt. Beim Übergang zu dem Falle $p>1$ wird die Variable z ersetzt durch eine komplexe Matrix

$$Z=X+iY=\|z_{hk}\|=\|x_{hk}+iy_{hk}\|$$

($z_{hk}=z_{kh}$; $h, k=1, 2, \cdots, p$) mit definit positiven Y. Die (geeignet definierten) analytischen Funktionen der Elemente z_{hk} der Matrix Z, die bei Substitutionen der beschränkten modularen Gruppe invariant bleiben, sind Abelsche Modular-Funktionen mehrer Variablen [Siegel, Math. Ann. 116 (1939), 617–657; MR 1, 203].

S. C. van Veen (Delft).

Lang, Serge. Algebraic groups over finite fields. Amer. J. Math. **78** (1956), 555–563.

Es sei k ein endlicher Körper mit q Elementen, und es sei G eine über k definierte algebraische Gruppe. [Für die Grundlagen der Theorie algebraischer Gruppen siehe Weil, Amer. J. Math. **77** (1955), 355–391, 493–512; MR **17**, 533.] Ist x ein Punkt von G, so bezeichne mann mit $x^{(q)}$ denjenigen Punkt von G, welcher entsteht, wenn man die Koordinaten von x mit q potenziert. Die Abbildung $f(x)=x^{-1}x^{(q)}$ ist eine rationale Abbildung von G in sich; Verf. zeigt, daß sie sogar eine Abbildung auf G ist. Bedeutet x einen allgemeinen Punkt von G, so ist die Körpererweiterung $k(x)/k(f(x))$ algebraisch, separabel und galoissch, wobei die galoissche Gruppe isomorph ist zu der Gruppe G_1 der in k rationalen Punkte von G.

Setzt man allgemeiner $F(x,y)=x^{-1}yx^{(q)}$, so erscheint dadurch G als homogener Darstellungsraum bezüglich G selbst. Unter Benutzung dieser Tatsache zeigt Verf., daß jeder homogene Raum H bezüglich G, welcher über k definiert ist, einen rationalen Punkt besitzt. Dieser Satz kann als Verallgemeinerung des vom Verf. früher bewiesenen Satzes angesehen werden, welcher folgendes besagt: Ist V eine über k definierte algebraische Mannigfaltigkeit, welche über der algebraisch-abgeschlossenen Hülle $\bar{k}$ von k biregulär äquivalent zu einer abelschen Mannigfaltigkeit ist, so besitzt V einen rationalen Punkt und ist also selbst eine abelsche Mannigfaltigkeit über k. [Siehe dazu Lang, Proc. Nat. Acad. Sci. **41** U.S.A. (1955), 174–176; MR **17**, 87.]

Weiter gibt Verf. einen neuen Beweis eines Resultates von Châtelet: Wenn eine algebraische Mannigfaltigkeit V/k biregulär äquivalent über $\bar{k}$ zum projektiven Raum ist, so ist sie bereits über k biregulär äquivalent zum projektiven Raum [C. R. Acad. Sci. Paris **224** (1947), 1616–1618; MR **9**, 55]. Beim Beweis benutzt Verf. den Satz, daß die einzigen biregulären Korrespondenzen des projektiven Raumes mit sich selbst die projektiven Abbildungen sind. Chow hat gezeigt, daß diese Eigenschaft außer dem projektiven Raum auch noch gewissen anderen Mannigfaltigkeiten zukommt [Ann. of Math. (2) **50** (1949), 32–67; MR **10**, 396]. Es ergibt sich daraus, daß auch der Satz von Châtelet für diese anderen Mannigfaltigkeiten gültig ist.

Schließlich betrachtet Verf. noch die Klassenkörpertheorie für die galoissche Erweiterung $k(x)/k(f(x))$. Es wird ein gewisses nicht-abelsches Reziprozitätsgesetz aufgestellt, und gezeigt, daß die zugehörigen Artinschen L-Reihen für Nichthauptcharaktere trivial sind. Hieraus wird das folgende Resultat gefolgert: Sei $\mathfrak{g}$ eine Untergruppe von G, welche nur aus rationalen Punkten über k besteht, und sei H der homogene Raum der Nebenklassen von G modulo $\mathfrak{g}$. Dann besitzen G und H die gleiche Anzahl von rationalen Punkten. In einer weiteren Arbeit [Ann. of Math. (2) **64** (1956), 285–325; MR **18**, 672] entwickelt Verf. eine abelsche Klassenkörpertheorie.

P. Roquette (Hamburg).

Denniston, R. H. F. On the topology of certain birational transformations. Ann. of Math. (2) **63** (1956), 10–14.

Dans ce mémoire, il s'agit de la détermination des modifications que subissent les groupes d'homologie d'une variété algébrique soumise à une transformation birationnelle. La dimension de la variété est quelconque, mais la transformation birationnelle est une dilatation au sens de Segre [Ann. Mat. Pura Appl. (4) **33** (1952), 5–48; MR **14**, 683]. Le résultat est atteint en adaptant à la situation un théorème de Chern et Spanier sur les fibrés.

Si $\mathfrak{M}$ est une variété algébrique non singulière, de dimension m, $\mathfrak{M}'$ son image dans la transformation birationnelle donnée, qui est une dilatation dont la base est la variété non singulière $\mathfrak{N}$, de dimension n, les nouveaux groupes d'homologie sont fournis à partir des données par la somme directe:

$$H_r(\mathfrak{M}'; I)\cong H_r(\mathfrak{M}; I)+\sum_{q=1}^{m-n-1} H_{r-2q}(\mathfrak{N}; I).$$

L. Gauthier (Nancy).

See also: van Spiegel, p. 162; Andreotti, p. 172; Atiyah, p. 172.

NUMERICAL ANALYSIS

Numerical Methods

Isida, Masatugu; and Ikeda, Hiroji. Random number generator. Ann. Inst. Statist. Math., Tokyo **8** (1956), 119–126.

A device for generating "random" numbers is presented. The paper stresses the logic of the system. By counting, during fixed time periods, the number X_t of output pulses of a G-M tube caused by radioactivity of Cobalt 60, the authors expect the counts to conform to a Poisson Distribution. When the mean of the distribution is large, say greater than 50, they contend that the "last figures of the X_t's are approximately equally distributed".

No data has been presented to support the claim that this device is satisfactory, a future paper promised to evaluate the equipment. It is somewhat surprising that the authors appear to be unaware of the fact that their device is not the first attempt to generate random numbers by counting pulses, for example Rand's approach [A million random digits with 100,000 normal deviates, Free Press, Glencoe, Ill., 1955; MR **16**, 749].

M. Muller (Princeton, N.J.).

Dimov, Lyubomir. Determination of best approximating circle by the method of least squares and by other methods. Jbuch. Staatsuniv. Stadt Stalin Fak. Bauwesen **1** (1953), 93–108. (Bulgarian. Russian summary)

Zmuda, A. J. Extrapolation of geomagnetic field components along a radius from the center of the earth. Trans. Amer. Geophys. Union **38** (1957), 306–307.

The author shows, by numerical examples, that the solid spherical harmonics of order up to 6 which represent the earth's geomagnetic field can be successfully extrapolated up to heights of 7 miles by assuming that they have the form $x(h)=x_0e^{-\beta h}$, where x_0 is x-component of the field at the surface of the earth, $x(h)$ the corresponding component at altitude h, and β is a coefficient found by numerical evaluation of the complete solid harmonics at two points near the ground and on the same radius.

J. A. O'Keefe (Chevy Chase, Md.).

Wilson, E. M. A family of integrals occurring in the theory of water waves. Quart. J. Mech. Appl. Math. **10** (1957), 244–253.

The author's summary states: An analysis is made of

integrals of the form

$$\int_0^{\frac{1}{2}\pi} e^{-\alpha^2\sec^2\theta} \sec^n \theta \, {}^{\cos}_{\sin} (\beta \sec \theta) d\theta$$

directed towards finding efficient methods of computation. Particularly useful results are differential equations in the β-direction and expansions in terms of Bessel functions which approximate to the integrals asymptotically.

J. C. P. Miller (Cambridge, England).

Saline, Lindon E. Quadratic programming of interdependent activities for optimum performance. Trans. A.S.M.E. **78** (1956), 37–46.

This paper proposes an algorithm for maximizing a quadratic function in r non-negative variables subject to m linear equations. The procedure is a generalization of the Simplex Method [Dantzig, Orden, and Wolfe, Pacific J. Math. **5** (1955), 183–195; MR **16**, 1054]. However, the mathematical discussion is deficient in that no attention is paid to the termination of the algorithm in a finite number of steps or to possible degeneracy in the constraining equations. An extensive numerical example illustrating the method is appended. *H. W. Kuhn.*

Morton, G.; and Land, A. H. A contribution to the "travelling-salesman" problem. (Symposium on linear programming.) J. Roy. Statist. Soc. Ser. B. **17** (1955), 185–194; discussion, 194–203.

This paper is concerned with finding the shortest closed path joining n points where the distances between all pairs of points are known. Several elementary properties are established for the case of a metric distance. A linear programming model and a computational technique replacing three links at a time are proposed. {Both conjectures were shown to be false by examples presented in the discussion following the symposium.}

H. W. Kuhn (Bryn Mawr, Pa.).

Zadunaisky, Pedro E. An iteration method for solution of systems of linear algebraic equations. Rev. Un. Mat. Argentina **17** (1955), 335–343 (1956). (Spanish)

The system of linear equations is written in the form $A_{11}x+A_{12}y=a_1$, $A_{21}x+A_{22}y=a_2$, with A_{ik} as matrix and with x, y, a_1, a_2 as column-vectors. The solution of such a system is obtained by iteration according to

$$x^{(n+1)}=A_{11}^{-1}[-A_{12}y^{(n)}+a_1],$$
$$y^{(n+1)}=A_{22}^{-1}[-A_{21}x^{(n)}+a_2],$$

and certain conditions for convergence of the sequences $x^{(n)}$, $y^{(n)}$ are given. In addition the author establishes a link from his iteration to what is known as the Gauss-Seidel iteration, especially in connection with the well-known paper by R. v. Mises and H. Pollaczek-Geiringer [Z. Angew. Math. Mech. **9** (1929), 58–77]. The reviewer would like to refer to the still more general method by Cesari and Picone, according to which the system $Ax=k$ is treated by the iteration $x^{(n+1)}=C^{-1}k+C^{-1}(C-A)x^{(n)}$; here A is a nonsingular matrix, k, x, are column-vectors, and C is an arbitrary nonsingular matrix, [see L. Cesari, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **25** (1937), 422–428]. *H. Bückner.*

Mendelsohn, N. S. An iterative method for the solution of linear equations based on the power method for proper vectors. Math. Tables Aids Comput. **11** (1957), 88–91.

It was pointed out by C. Lanczos [J. Res. Nat. Bur. Standards **49** (1952), 33–53; MR **14**, 501] that solving a system of linear algebraic equations, (1) $Ax=b$, is equivalent to determining the eigenvector corresponding to the largest eigenvalue of a matrix related to the given matrix A as follows. Suppose A is an $n\times n$ matrix and b is an n-vector. Form the $(n+1)\times(n+1)$ matrix,

$$A_1=\begin{pmatrix} A & -b \\ 0 & 0 \end{pmatrix},$$

and let $y=(x, 1)$. Then (1) is equivalent to (2) $A_1y=0$. (2) can be regarded as the problem of minimizing $(A_1y)^2$ under the condition $y^2=1$, which is equivalent to finding the solution of the eigenvalue problem (3) $A_1{}^*A_1y=\lambda y$ for the eigenvalue $\lambda=0$. Let λ_M be an upper bound for the eigenvalues of $A_1{}^*A_1$ as obtained, for example, by Gerschgorin's method. Then Lanczos sets $A_2=\lambda_M I-A_1{}^*A_1$ and uses a modification of the power method to obtain the eigenvector corresponding to the largest eigenvalue of A_2. This solves (3) for the smallest eigenvalue, $\lambda=0$.

In the paper under review, the author forms the matrix,

$$B=\begin{pmatrix} A+kI & -b \\ 0 & k \end{pmatrix}$$

and writes (1) as (4) $By=ky$. Thus, y is an eigenvector of B corresponding to the eigenvalue, k, where k is any real constant. However, since the power method is used to find y, k must be larger in modulus than any other eigenvalue of B. In this connection the author proves the following theorem. Let A be non-singular. A necessary and sufficient condition that there exist a real number k such that each eigenvalue of B distinct from k be of modulus less than $|k|$ is that all eigenvalues of A have non-zero real parts of the same sign.

As a practical expedient, the author suggests choosing k to be a large multiple of $-\mathrm{tr}\, A$. If the power method does not converge, he suggests solving the system, $A^*Ax=A^*b$, for which his theorem is valid. *E. K. Blum.*

Altman, M. On the solution of linear algebraic equations. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 93–97, IX. (Russian summary).

It is shown that most of the common methods of solving linear algebraic equations can be deduced from a more general biorthogonalisation process. A related orthogonalisation method is given, with explicit formulae for the solution. *L. Fox* (Teddington).

Akušskiĭ, I. Ya. On conditions of solvability of an homogeneous computational problem. Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. **1956**, no. 4(8), 112–127. (Russian)

The problem referred to in the title is simply that of calculating a sequence of vectors $y_1, y_2, \cdots$ defined in terms of a given vector y_0 and a given matrix U by $y_k=U^ky_0$. By "solving" this problem the author means computing y_k by an indirect process whenever the machine is unable to accomodate multiplication directly by U. There is constructed a certain cycle of simpler operations achieving multiplication by U or a power of U. Certain conditions on such a program and its application are discussed in great detail. Although no particular machine is mentioned, a possible example is a punched card tabulator or relay multiplier. *D. H. Lehmer.*

Akušskiĭ, I. Ya. Certain questions of improvement of the solvability of a homogeneous computation problem. Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. **1956**, no. 5(9), 71–89. (Russian)

Akušskiĭ, I. Ya. On solvability of the inverse matrix. Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. **1956**, no. 5(9), 90–100. (Russian)

These are the second and third of a series of papers on

the use of operational matrices to solve various linear problems of finite matrices. The first paper [see the paper listed above] is said to have the essential purposes and definitions; without that paper the reviewer feels somewhat uncertain. More papers are announced.

An "operational matrix" Ω is the product of some finite number r of square matrices of form $I+E_{ij}$, where I is the identity and

$$E_{ij}=\begin{cases}1 \text{ or } -1 \text{ in}(i,j)\text{-th element,}\\ 0 \text{ elsewhere.}\end{cases}$$

Note that $(I+E_{ij})^{-1}=I-E_{ij}$. Pre- or post-multiplication by Ω or Ω^{-1} is computationally very easy, because of the simple structure.

Given a (row) vector Y_0, and square matrix U, and integer j. Let $Y_i=U^iY_0$. A simple but typical method of the author is to find a (column) vector A_0 and an operational matrix Ω such that the jth component of $A_i=\Omega^iA_0$ is equal to the jth component of Y_i. {Such a substitution of Ω for U could prove useful in computing eigenvalues, either by direct or inverse iteration.}

On pp. 71–73, the author greatly generalizes the above method to involve two-sided matrix multiplication and non-homogeneous operations, and states the general condition for finding matrices analogous to Ω. He then lists some 15 special cases covering most iterative matrix processes.

The last part of the same paper considers the augmentation of the n-by-n matrix U to an $(n+l)$-by-$(n+l)$ matrix $\mathbf{U}$ by adding l rows and l columns to U. A theorem tells all ways to do this so that the n-by-n upper-left-hand principal minor of $\mathbf{U}^k$ is U^k, for $k=0, 1, 2, \cdots$.

The next paper is devoted mainly to the use of Ω^{-1} for inverse iteration by U — i.e., for computing components of Y_i, where

$$Y_{i+1}=U^{-1}Y_i+d,$$

where d may be 0.

Both papers have some numerical examples of orders 2 and 3. *G. E. Forsythe* (Stanford, Calif.).

Meyer, H. I.; and Hollingsworth, B. J. A method of inverting large matrices of special form. Math. Tables Aids Comput. 11 (1957), 94–97.

A method is given for inverting large matrices of special form, such as arise in the numerical solution of Laplace's equation and similar problems. The large matrix has the appearance of a symmetric co-diagonal form, but the elements are themselves square matrices of smaller order. The computations involve inversion and multiplication of matrices of this order, and convenient recurrence relations are produced of advantage in interpretive programming for a high-speed computer. *L. Fox* (Teddington).

Doyle, Thomas C. Inversion of symmetric coefficient matrix of positive-definite quadratic form. Math. Tables Aids Comput. 11 (1957), 55–58.

The $n\times n$ symmetric positive definite matrix a is associated with the positive definite quadratic form, x^Tax, with x real. The inverse of a is determined with the use of the upper triangular matrix H having the properties $|H|=1$ and $H^TaH=d^{-1}$, where d^{-1} is diagonal. Then $a^{-1}=HdH^T$, $|a|=|d^{-1}|$, and $x^Tax=\xi^Td^{-1}\xi$, where $x=H\xi$. Then the necessary and sufficient condition that x^Tax be positive definite is that d_i^{-1} be positive for each i.

The method of constructing H is described for $n=4$. The matrix H_i is an identity matrix with the post-diagonal portion of its ith row replaced by elements obtained in the ith Gaussian elimination. Then $H=H_1H_2H_3$. The values of d_i^{-1} are determinantal ratios. Then a^{-1} is obtained from H and d. The method has been coded for the IBM 704 at Los Alamos. The results give the solution of $ax=b$, the values of d_i^{-1}, and $|a|$, as well as the elements of the inverse matrix. Timing and accuracy descriptions are included. *P. S. Dwyer* (Ann Arbor, Mich.).

Akušskiĭ, I. Ya. Some properties of matrices that reflect the steps and operations on computers. Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. **1956**, no. 4(8), 86–100. (Russian)

The inversion and raising to powers of matrices with a punched card tabulator is discussed in general terms and illustrated in the case of an 8-counter tabulator.

D. H. Lehmer (Berkeley, Calif.).

Hart, William L.; and Motzkin, Theodore S. A composite Newton-Raphson gradient method for the solution of systems of equations. Pacific J. Math. 6 (1956), 691–707.

The authors present an iterative method for the solution of a fairly general system of k equations (1) $f_j(x)=0$ $(j=1, \cdots, k)$ in n unknowns $(x_1, \cdots, x_n)=x$, where all variables and functions are real-valued. The iteration process consists in choosing an initial approximation, $x^{(0)}$, and determining a sequence $\{x^{(m)}\}$ according to the formula (2) $x^{(m)}=x^{(m-1)}+\rho^{(m-1)}\Delta x^{(m-1)}$, where the $\rho^{(m)}$ are positive numbers and the $\Delta x^{(m)}$ are vectors defined as follows: Let the partial derivatives $f_{ij}=\partial f_j/\partial x_i$ be continuous in some open convex region Ω in n-space. Let $x^{(m-1)}$ and $\xi^{(m-1)}$ be points in Ω and define

$$(3) \qquad \Delta_j x_i^{(m-1)}=-f_j(x^{(m-1)})f_{ij}(\xi^{(m-1)})/W_j^2(\xi^{(m-1)}),$$

where $W_j^2(x)=\sum_{i=1}^n f_{ij}^2(x)$. Using (3), form the vectors (4) $\Delta_j x^{(m-1)}=(\Delta_j x_1^{(m-1)}, \cdots, \Delta_j x_n^{(m-1)})$. Choose an arbitrary set of positive "weights", $(\eta_1, \cdots, \eta_k)$ and define

$$(5) \qquad \Delta x^{(m-1)}=\sum_{j=1}^{n}\eta_j\Delta_j x^{(m-1)}.$$

Formula (3) is of the Newton-Raphson type and the vectors (4) are parallel to the gradients of the f_j. To this extent the method is similar to other iterative gradient methods. As the authors point out, in the linear case with (6) $f_j(x)\equiv\sum_{h=1}^n x_ha_{hj}+b_j=0$, the method is equivalent to minimizing the function, $g(x)=\sum_{j=1}^n \eta_jf_j^2(x)$, by applying corrections $\rho\Delta x=\text{grad}\,[\frac{1}{2}\rho g(x)]$.

For the linear system (6), conditions are obtained for the convergence of $\{x^{(m)}\}$ to the point $\hat{x}$ nearest $x^{(0)}$ in the set ψ of all points where $g(x)$ attains its absolute minimum. Let $\zeta_j=\eta_j^{\frac{1}{2}}/(\sum_{i=1}^n a_{ij}^2)^{\frac{1}{2}}$ and scale the matrix by writing $\alpha_{hj}=\zeta_ja_{hj}$. Denote the matrix (α_{hj}) by B and let $\lambda_1, \cdots, \lambda_r$ be the positive eigenvalues of BB'; i.e., r is the rank of (a_{hj}). In (2), set $\rho^{(m)}=\rho$ for all $m\geqq0$ and define $\sigma_\rho=\max_{i\geqq r}|\mu_i|$, where $\mu_i=1-\rho\lambda_i$. A necessary and sufficient condition that $\{z^{(m)}\}$ should converge for arbitrary $x^{(0)}$ is that $\sigma_\rho<1$. If $\sigma_\rho<1$, then (7) $\|x^{(m)}-\hat{x}\|\leqq\sigma_\rho^m\|x^{(0)}-\hat{x}\|$. A sufficient condition that $\sigma_\rho<1$ is that $0<\rho<2/w$, where $w=\sum_{j=1}^k \eta_j$. The minimum value of σ_ρ occurs for a unique value $\rho=\rho_0$, where $2/w<\rho_0<2(r-1)/w$ if $r>2$, and $\rho_0=r/w$ if $r\leqq2$.

For the general system (1) the method requires an initial guess $x^{(0)}$ which is close to the solution $\hat{x}$. The following theorem is obtained: Assume that (1) has the solution $\hat{x}$ in Ω, that the Jacobian (f_{ij}) evaluated at $\hat{x}$ is of rank n, and $W_j^2(x)\neq0$ for x in Ω. If the $\rho^{(m)}$ are

properly chosen, with $0<\rho^{(m)}<2\eta/w$, there exists a $\theta>0$, with $\theta<1$, and a corresponding $\delta>0$ such that if $x^{(0)}$ and $\xi^{(m)}$ are arbitrary points in the neighborhood $\|x-\hat{x}\|<\delta$, then $x^{(m)}$ converges to $\hat{x}$ and $\|x^{(m)}-\hat{x}\|\leq\theta^m\|x^{(0)}-\hat{x}\|$. It is sufficient that $\rho^{(m)}=\rho$, where $\rho\leq 2/w$ if $n>1$ and $\rho<2/w$ if $n=1$.

The remainder of the paper concerns the application of the method to the determination of the implicit function $x=x(t)$ defined by the system $f_j(x,\tau)=0$, where $\tau=(\tau_1,\cdots,\tau_k)$ and $\tau=\tau(t)$, $0\leq t\leq 1$. The technique employed is to choose a partition $0=t_0<t_1<\cdots<t_q=1$ sufficiently fine that $\{x^{(m)}\}$ will converge to $x(t_i)$ if $x^{(0)}=x(t_{i-1})$.

No numerical examples are given. *E. K. Blum.*

Lebedev, A. N. Theory of automatization of the solution of a system of two algebraic equations. Leningrad. Èlektrotehn. Inst. Izv. **1953**, no. 25, 28–39. (Russian)

Bauer, Friedrich L.; und Samelson, Klaus. Polynomkerne und Iterationsverfahren. Math. Z. **67** (1957), 93–98.

Recent investigations by the senior author and others [e.g., F. L. Bauer, S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1954**, 275–303; MR **18**, 151; H. Rutishauser, Z. Angew. Math. Phys. **5** (1954), 233–251; MR **16**, 176] have revealed a wide class of iterative methods related to that of Bernoulli, for computing the zeros of a polynomial. The present paper exhibits four, but omits convergence proofs. To give one, if $P(x)$ is of degree n with leading coefficient unity, $q^{(0)}(x)$ is of degree $n-1$ at most, and $\beta^{(0)}$ is in a suitable neighborhood of a zero λ of $P(x)$, then the recursions

$$q^{(i+1)}(x)=(x-\beta^{(i)})^{-1}[P(x)-P(\beta^{(i)})q^{(i)}(x)/q^{(i)}(\beta^{(i)})],$$
$$\beta^{(i+1)}=\beta^{(i)}-P(\beta^{(i)})/q^{(i)}(\beta^{(i)})$$

define sequences of polynomials and constants which converge quadratically to $P(x)/(x-\lambda)$ and to λ, respectively. *A. S. Householder* (Oak Ridge, Tenn.).

Head, J. W. Widening the applicability of Lin's iteration process for determining quadratic factors of polynomials. Quart. J. Mech. Appl. Math. **10** (1957), 122–128.

This paper discusses the application of Aitken's process for accelerating the convergence of a nearly geometric sequence, to Lin's process for determining quadratic factors of polynomials. *J. C. P. Miller.*

Ghéorghiev, Ghéorghi Iv. Formules de quadrature mécanique des polynômes de deux variables réelles. Univ. d'État Varna "Kiril Slavianobâlgarshi". Fac. Tech. Méc. Annuaire 3 (1947–1948), 1–46 (1949). (Bulgarian summary)

La classe C est composé de certaines fonctions $f(x,y)$ des variables réelles x et y, pour chacune desquelles l'intégrale double $\iint f(x,y)dxdy$ prise sur le domaine R du plan XOY est complètement déterminée. L'auteur cherche à déterminer le nombre natural N dans la formule

$$(1)\qquad \iint_{(R)} f(x,y)dxdy=\sum_{m=1}^{N}\lambda_m f(x_m,y_m)$$

de telle manière qu'étant pris tout autre nombre naturel $N_1<N$ il existe toujours au moins une fonction $\varphi(x,y)$ de la classe C telle que pour $f(x,y)=\varphi(x,y)$ l'égalité (1) soit en défaut, quelle que soit la manière dont on a choisi les nombres λ_m, x_m, y_m dans la formule (1), où l'on a remplacé N par N_1. Au cas où existent des formules de l'espèce (1), se pose le problème de leur détermination.

Dans le travail présent l'auteur donne la solution du problème ainsi posé, de quadrature mécaniques de quelques classes C_n de polynomes des variables réelles x et y dont le degré ne dépasse pas n. L'auteur détermine pour les cas $n=1$, 2 toutes les formules de quadrature mécanique par rapport à un domaine arbitraire R. Pour le cas $n=1$ il trouve une seule formule à un terme ($N=1$). Pour le cas $n=2$ il trouve un nombre infini de formules à trois termes. ($N=3$). Pour le cas $n=3$ l'auteur se restreint à un domaine R symétrique par rapport à un point. Il trouve alors un nombre infini de formules à quatre termes ($N=4$). *S. C. van Veen* (Delft).

Hammer, P. C.; Marlowe, O. J.; and Stroud, A. H. Numerical integration over simplexes and cones. Math. Tables Aids Comput. **10** (1956), 130–137.

Let $\xi=(\xi_1,\xi_2,\cdots,\xi_n)$ be a point and R a region of E_n. Let C be the cone $x\xi$ ($\xi\in R$; $0\leq x\leq 1$) of the space E_{n+1} with the points $(\xi_1,\xi_2,\cdots,\xi_n;x)$. It is aimed to evaluate an integral $J=\int_C f(\xi,x)\,d\xi dx$. The authors write

$$J=\int_0^1 dxx^n\int_R f(x\xi,x)d\xi\approx J^*$$

with $J^*=\int_0^1 x^n(gx)dx$; $g(x)=\sum_{j=1}^m a_j f(x\xi^{(j)},x)$ expresses a formula of numerical integration over the n-dimensional region R. The integral J^* itself is evaluated by a formula $J^*\approx J^{**}=\sum_{i=1}^l b_i g(x_i)$. The combination of the rules $g(x)$ and J^{**} gives an integration formula for $(n+1)$-dimensional cones. This procedure permits stepping up from integration formulas for line intervals to formulas for a polyhedron in hyperspace. Special emphasis is on formulas which hold exactly for polynomials f; the formula J^{**} yields the importance of Jacobi-polynomials. Preliminary results on symmetrical formulas are presented. These refer to the invariance of weights under affine transformations of a region onto itself. The paper also contains an example and some tables.

H. Bückner (Schenectady, N.Y.).

Hammer, Preston C.; and Stroud, Arthur H. Numerical integration over simplexes. Math. Tables Aids Comput. **10** (1956), 137–139.

With reference to the paper reviewed above, two formulas of integration of polynomials over n-dimensional simplices are presented. The first one represents a weighted sum over the integrand at the vertices and the centroid; it is exact for cubic polynomials. The second one uses the vertices only and is exact for quadratic polynomials. *H. Bückner* (Schenectady, N.Y.).

Fishman, Herbert. Numerical integration constants. Math. Tables Aids Comput. **11** (1957), 1–9.

The author presents in this paper a list of orthogonal polynomials associated with $\int_0^1 x^n f(x)dx$ with degrees 1 to 8 and the related constants b_j, x_j for the numerical integration sums $\sum_m^1 b_j f(x_j)$ $n=0(1)5$, $m=1(1)8$, to 12D. The table extends those given by Hammer, Marlowe and Stroud in the paper reviewed second above. Uses of these tables may be made in developing numerical integration formulas for cones and hypercones with dimensions through 9 when a formula for the base is at hand. As a special application numerical values of iterated integrals may be found. For $n=0$ the Gaussian quadrature formula results. *P. C. Hammer* (Madison, Wis.).

Sharma, A. On Golab's contribution to Simpson's formula. Ann. Polon. Math. 3 (1957), 240–246.

The problem of the quadrature of a function $f(x)$ by means of a rule

$$P(h)=\int_a^{a+h} f dx \approx \bar{P}(h)=h[p_0 f(a)+p_1 f(a+\theta_1 h)+p_2 f(a+\theta_2 h)]$$

with $0<\theta_1\leq\frac{1}{2}<\theta_2\leq 1$ is studied under the assumption that $f(a+h)=f(a)+a_p h^p+a_q h^q+a_r h^r+O(h^s)$ $(1\leq p<q<r<s)$, where p, q, r, s are positive integers, and $a_p, a_q, a_r\neq 0$. Under the condition of fixed values θ_1, θ_2 the author looks for the best weights p_0, p_1, p_2 in the sense that $P(h)-\bar{P}(h)$ is of the highest order of smallness with respect to h. He finds that there is exactly one solution; he gives formulas for the weights, which relate them to θ_1, θ_2, p, q. The error is of the order h^{r+1}. Next the same problem with respect to the best values of the weights and of θ_2 is discussed. It is found that this problem has a unique solution and that the order of the error is h^{s+1}.

H. Bückner (Schenectady, N.Y.).

Miller, J. C. P. Note on the general solution of the confluent hypergeometric equation. Math. Tables Aids Comput. 11 (1957), 97–99.

This paper gives a full discussion of the complete solution of the equation $xy''+(\gamma-x)y'-\alpha y=0$ in certain degenerate situations which are often overlooked. [For references to earlier work see M. S. Corrington, same journal 2 (1947), 352–353].

John Todd (Pasadena, Calif.).

Edelman, Franz. An interpretive subroutine for the solution of systems of first order ordinary differential equations on the 650. IBM Appl. Sci. Tech. Newsletter no. 13 (1957), 52–72.

For solving the system of equations

$$y_i'=f_i(y_0, y_1, y_2, \cdots, y_N),$$

with initial conditions $y_i(y_{00})=y_{i0}$ $(i=1, 2, \cdots, N)$, the routine described offers the user the choice of using a Runge-Kutta-Gill (RKG) integration procedure, or a Milne predictor-corrector formula, both methods being correct to terms of h^4 where h is the interval of integration. The Milne methods employ the RKG for starting values. In it the precision may be set by specifying that the predictor and corrector are to agree either for a given number of digits or for a given number of significant figures. When the precision becomes too large, h is automatically doubled, and when it is too small, h is reduced by a preset factor, usually 0.5.

The program allows for N to be as large as 49 and it provides a very flexible type and amount of output. The execution time per point is about $(6+3N)$ seconds for the RKG method and $(2.5+1.5N)$ seconds for the Milne method, plus the time required for evaluating the functions f_i. Since the overall logic of the program is applicable to machines other than the IBM 650 the inclusion of the flow diagram for the program is valuable.

C. C. Gotlieb (Toronto, Ont.).

Ridley, E. Cicely. A numerical method of solving second-order linear differential equations with two-point boundary conditions. Proc. Cambridge Philos. Soc. 53 (1957), 442–447.

This paper gives a method for solving second-order linear differential equations with two-point boundary conditions by factorizing the differential equation. This method is shown to be closely related to the matrix factorization method of Fox and Thomas, and has similar stability conditions. The method seems to be well suited to hand computation, or hand computation supplemented by an automatic computer, but not to be well suited to a large computer without human intervention now and then.

R. W. Hamming (Murray Hill, N.J.).

Viswanathan, R. V. Solution of Poisson's equation by relaxation method—normal gradient specified on curved boundaries. Math. Tables Aids Comput. 11 (1957), 67–78.

This paper attempts to solve the difficult problem of finding adequate finite-difference approximations to the Laplace operator in the neighbourhood of a curved boundary on which the function has a known normal derivative. The formulae obtained involve at most five internal values, at nodes of a square mesh, but the coefficients are somewhat complicated and require prior evaluation of the radius of curvature of the boundary curve at an adjacent point, the angle between the normal at this point and one of the coordinate axes, and the rate of change of the given normal derivative in the tangential direction. The error in the formulae is of order h^3, where h is the mesh length, and this is an improvement on previous methods. A numerical example illustrates the application of the formulae.

L. Fox (Teddington).

Halilov, Z. I. Solution of a problem for an equation of mixed type by the method of nets. Dokl. Akad. Nauk Azerbaĭdžan. SSR. 9 (1953), 189–194. (Russian. Azerbaijani summary)

For the equation

$$\frac{\partial^2 u}{\partial x^2}+\theta(y)\frac{\partial^2 u}{\partial y^2}=0,$$

where $\theta(y)=1$ for $y>0$, $\theta(y)=-1$ for $y<0$, in a domain D bounded in the upper half-plane by the curve L with endpoints $A=(0, 0)$ and $B=(1, 0)$ and in the lower half-plane by segments L_1 and L_2 of characteristics, the author considers the problem of finding a solution which satisfies the conditions

$$u|_L=\phi, \quad u|_{L_1}=\psi.$$

This problem is solved by the method of nets.

The domain of ellipticity of the given equation is divided, by straight lines parallel to the x and y axes, into squares of sides $h=1/n$ and the given equation is replaced by the condition that at interior points of the lattice the value of u_h is equal to the arishmetic mean of the values of u_h at the four neighboring vertices. At the boundary points of the lattice adjacent to L, u_h is set equal to ϕ.

The domain of hyperbolicity $(y<0)$ is divided into squares by straight lines inclined to the axis Ox at an angle of $\pm\pi/4$ and issuing from the points $(m/n, 0)$, $m=0, 1, \cdots, n$. In this case the given equation is replaced by the relation

$$u_h(x, y)-u_h\left(x+\frac{1}{2n}, y-\frac{1}{2n}\right)=u_h\left(x-\frac{1}{2n}, y-\frac{1}{2n}\right)-u_h\left(x, y-\frac{1}{n}\right).$$

Finally, at points of the lattice belonging to AB, the condition is

$$u_h\left(\frac{m}{n}, 0\right)=\frac{1}{2}\left[u_h\left(\frac{m}{n}, \frac{1}{n}\right)+u_h\left(\frac{m}{n}, -\frac{1}{n}\right)\right],$$

and at points of L_1, $u_h=\psi$.

The author asserts that, for sufficiently small h, the u_h so obtained will differ arbitrarily little from the actual value of u, provided the latter exists and is four times differentiable. For the solution of the algebraic system determining u_h, a method of successive approximations is indicated which is similar to the balayage method in the Dirichlet problem.
O. A. Ladyženskaya (RŽMat 1954, no. 1401).

Budak, B. M. On the method of straight lines for certain boundary problems. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. 11 (1956), no. 1, 3–12. (Russian)

The author studies the error committed by approximating the solution $u(x, y)$ of the Goursat problem

$$u_{xy}=a(x, y)u+f(x, y)\ \ (0\leq x\leq l';\ 0\leq y\leq l''),$$
$$u(0, y)=\rho(y)\ \ (0\leq y\leq l''),$$
$$u(x, 0)=\psi(x)\ \ (0\leq x\leq l')$$
$$\varphi(0)=\psi(0)$$

on the lines $x=x_k=kh$ ($k=1, 2, \cdots, n$; $nh=l'$) by the solutions $u_k(y)$ of the following recursive system of ordinary differential equations ($k=1, 2, \cdots, n$; $0\leq y\leq l''$):

$$u_{k+1}'(y)=u_k'(y)+h[a(x_k, y)u_k(y)+f(x_k, y)],$$
$$u_k(0)=\psi(x_k),\quad u_0(y)=\varphi(y).$$

It is shown that if φ', ψ', f, f_x, a, and a_x are continuous,

$$|u_k(y)-u(x_k, y)|\leq Ch\ \ (k=1, 2, \cdots, n;\ 0\leq y\leq l''),$$

where the constant C is expressed explicitly in terms of the data of the problem. The numerical work is equivalent to that of one step of the method of successive approximations. Similar error estimates (also showing degree of convergence 1) are obtained for the application of the method to the Goursat problem for the non-linear equation

$$u_{xy}=A(x, y, u)$$

and to the classical boundary problems for the equations

$$\rho(x)Du=(\delta(x)u_x)_x-q(x)u+f(x, t),$$

where Du is either u_t or u_{tt}, and ρ and δ are positive and bounded away from zero. In the latter case the lines are chosen parallel to the t-axis; no stability considerations are required. The method is also applied to systems of equations of the above types. In the case of a system of telegraph equations the degree of convergence is only $\frac{1}{2}$. {Reviewer's note: It seems that the degree of convergence of the method could be improved by using symmetric rather than one-sided approximations to first derivatives.}
P. Henrici (Los Angeles, Calif.).

Rašba, E. I. Treatment of points near the contour in using finite difference equations approximating the biharmonic differential equation. Akad. Nauk Ukrain. RSR. Prikl. Meh. 2 (1956), 210–216. (Ukrainian. Russian summary)

The finite difference equation used in the computation of approximate solutions of a biharmonic equation involves for each point the values of the function in 12 neighboring points of the net. For points near the boundary some of these neighboring points lie outside the region; different authors use different interpolations for such points involving the value of the function in one point of the region and the value of the normal derivative in a boundary point. The author proposes to use a more exact interpolation involving the values at 4 points of the region, and gives a numerical example in which the exact solution is known and in which his method is seen to give a much better approximation than two other methods.
G. Y. Rainich (Ann Arbor, Mich.).

Mann, W. Robert; Bradshaw, C. L.; and Cox, J. Grady. Improved approximations to differential equations by difference equations. J. Math. Phys. 35 (1957), 408–415.

The authors show that the truncation error in approximating a differential equation by a difference equation can be reduced by modifying the coefficients in the difference equation from those usually used. The modified coefficients are found by expanding the usual difference equation in a Taylor series and inspecting the difference between the expansion and its leading terms, which constitute the left-hand side of the differential equation. By judiciously expressing this error in terms of differences it is possible to incorporate part of it with the usual difference equation in such a way as to substantially reduce the truncation error without increasing the order of the difference equation.
E. Pinney (Berkeley, Calif.).

Korolyuk, V. S. On a method of increasing the asymptotic exactness of the method of grids. Ukrain. Mat. Ž. 7 (1955), 379–387. (Russian)

★ **Wielandt, Helmut. Error bounds for eigenvalues of symmetric integral equations.** Proceedings of Symposia in Applied Mathematics. Vol. VI. Numerical analysis, pp. 261–282. Published by McGraw-Hill Book Company, Inc., New York, 1956 for the American Mathematical Society, Providence, R. I. $9.75.

The oldest method of treating the problem

$$\int_0^1 K(s, t)y(t)dt=\kappa y(s)$$

in a numerical way refers to a rule S of numerical quadrature, $Sf=\sum_{k=1}^n p_kf(x_k)$ for the integral $\int_0^1 f(x)dx$. Applied to the integral-equation it leads to the system $\sum_{k=1}^n p_kK(x_i, x_k)y_k=\kappa' y_i$ of simultaneous equations, the nontrivial solutions of which approximate the eigenelements of the integral equation in the sense $\kappa'\approx\kappa$; $y_k\approx y(x_k)$. The author proposes to derive error bounds for the approximations to the eigenvalues κ of a square integrable and Hermitian kernel $K(s, t)$. The abscissas are in the interval $\langle 0, 1\rangle$; the weights p_k are positive and their sum is unity. The eigenvalues κ are ordered in the sense $\kappa_1\geq\kappa_2\geq\cdots\to 0$; $\kappa_{-1}\leq\kappa_{-2}\cdots\to 0$; the same ordering applies to the approximations κ'. The same subscript establishes the relation of exact value and approximation. The error bounds are of the type $|\kappa-\kappa'|\leq M$ with M valid for all approximations simultaneously. As a basic approach to results of this type the author introduces the concept of a Hermitian kernel $G(s, t)$ allowing rule S; this means that the rule furnishes exact eigenvalues of G. As a special result it is proved that any rule S applied to K admits the error bound $|\kappa-\kappa'|\leq\|K-G\|$ with the symbol $\|H\|$ denoting the largest modulus of the eigenvalues of H. The bound holds for any G allowing S and coinciding with K at the meshpoints $s=x_i$, $t=x_k$. For practical applications G is constructed as a degenerate kernel. Furthermore $\|K-G\|$ is replaced by $\sup_{s,t}|K-G|\geq\|K-G\|$ or by the number $\|P\|$ of a degenerate Hermitian kernel $P(s, t)\geq|K-G|$. A typical example refers to equal weights and to the abscissas $x_k=(k-1)/n$. Assuming a Lipschitz-

condition $|K(s,t)-K(x_i,x_k)|\leq L[p(s)+p(t)]$ with $p(s)=s-x_k$ for $x_k\leq s<x_{k+1}$ the author finds $|\kappa-\kappa'|\leq C\cdot L/n$; $C=\frac{1}{2}+\sqrt{\frac{1}{3}}$. This holds for all kernels K with the Lipschitz-constant L; the constant C cannot be replaced by a smaller one. In similar vein the central point rule, the trapezoidal rule, Simpson's and Gauss's rule are investigated. All of this is illustrated by numerical examples. Limitations of the method as well as directions of research for refining the bounds are discussed at the end of the paper. It is pointed out that smaller bounds should be found with respect to a portion of the approximate eigenvalues, say the first ones; thus error bound theorems would cover a smaller number of approximate eigenvalues but also admit smaller bounds in general.

H. Bückner (Schenectady, N.Y.).

★ **Warschawski, S. E. Recent results in numerical methods of conformal mapping.** Proceedings of Symposia in Applied Mathematics. Vol. VI. Numerical analysis, pp. 219–250. Published by McGraw-Hill Book Company, Inc., New York, 1956 for the American Mathematical Society, Providence, R.I. $9.75.

Four methods of establishing boundary relations of conformal mapping are discussed from a numerical point of view. These are related to the problem of mapping a simply connected region of the z-plane, bounded by a closed Jordan curve C, onto the unit circle $|w|\leq 1$ of the w-plane. The first method is based on the Lichtenstein-Gershgorin integral equation

$$\theta(s)=\int_0^L K(s,t)\theta(t)dt-2\beta(s),\quad \theta(s)=\arg f(\zeta(s)),$$
$$K(s,t)=\partial \log r_{st}/\partial n_t$$
$$r=|\rho(s)-\rho(t)|,\ \beta(s)=\arg[(z_1-\rho(s))(z_0-\rho(s))^{-1}],$$

with $w=f(z)$ denoting the mapping function and $z=\zeta(s)$ defining C with respect to the arclength s of C; z_0, z_1 correspond to $w=0$, $w=1$ respectively. The inner normal at point $\zeta(t)$ is n_t. The kernel K is known from Dirichlet's problem for the exterior of C; its eigenvalues are $\lambda_1=1$, $\lambda_2,\cdots$ in the order of increasing moduli; the resolvent of K has the principal part $\mu(t)(1-\lambda)^{-1}$ at the pole $\lambda=\lambda_1=1$. The function β is orthogonal to μ, and the integral equation admits solutions. Starting with a continuously differentiable function $\theta_0(s)$, a sequence of iterations $\theta_{n+1}(s)=\int_0^L K(s,t)\theta_n(t)dt-2\beta(s)$ is calculated. It is proved that the sequence converges uniformly to that solution θ, for which $\theta-\theta_0$ is orthogonal to μ; here it is assumed that $\zeta'(s)$ satisfies a Hoelder condition of order α; $0\leq\alpha<1$. The same holds for the derivatives $\theta_n'(s)$ if $\zeta''(s)$ satisfies that Hoelder condition. Moreover bounds for the error $|\theta-\theta_n|$ are given. These imply $|\theta-\theta_n|=O(|\lambda_2|^{-n})$; the rate of convergence depends on $|\lambda_2|$; therefore some estimates of $|\lambda_2|$ are presented. These relate λ_2 to the known λ_{20} of a contour C_0, which is close to C in a specified sense. Estimates for nearly circular and nearly convex contours are obtained. For ellipses of various sizes computational experiments at the Computation Laboratory of NBS in Washington under the direction of J. Todd [Todd and Warschawski, Nat. Bur. Standards Appl. Math. Ser. no. 42 (1955), 31–44; MR 17, 540] were carried out. The integral operation was replaced by Weddle's rule with respect to the parameter τ of the representation $x=a\cos\tau$; $y=b\sin\tau$ of the ellipse C. It was found that Aitken's δ^2-process was helpful in order to reduce the number of iterations. A nine-place accuracy was aimed at. Tables of these iterations are included in the paper.

The second method applies to nearly circular regions and is due to M. S. Friberg [Thesis, Univ. of Minnesota, 1951]. C is represented in polar coordinates ρ, ϕ with $\rho=\rho(\phi)$ assumed to be continuously differentiable up to second order. The angles ϕ, θ are related by the integro-differential equation

$$\phi'(\theta)=\cos\beta(\phi)\cdot\exp\Big[(2\pi)^{-1}\int_0^\pi\{\beta[\phi(\theta+t)]-\beta[\phi(\theta-t)]\}\cot\tfrac{1}{2}t\,dt\Big]=T(\phi(\beta))$$

with $\beta=-\operatorname{arctg}(\rho^{-1}d\rho/d\phi)$. The iteration $\phi_{n+1}'=T(\phi_n)$ together with the condition $\int_{-\pi}^{\pi}\phi d\theta=0$ is used; numerically the integrations are carried out by replacing ϕ_n by a trigonometric polynomial in θ, which coincides with ϕ_n at equidistant points of the θ-interval. ϕ can be obtained with prescribed accuracy by iteration if the pivotal points are sufficiently dense. A numerical test was carried out at the University of Minnesota. The paper gives a summary of these calculations.

The third method deals with approximating the mapping function $f(z)$ by polynomials which are found by the Rayleigh-Ritz-method. With $f(z_0)=0$, $f'(z_0)=1$ prescribed, the variational problem $\int_C|F(z)|^2ds=$minimum; $F(z_0)=1$; $F(z)$ of a certain class $H_2(C)$ has the solution $F=(f')^{\frac{1}{2}}$. The solution in general as well as in the case, where the competing functions F are polynomials of order $\leq n$ is known from Szegö's work on orthogonal polynomials [Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ., v. 23, New York, 1939, p. 359; MR 1, 14]. However the author adds an estimate of the degree of approximation by polynomials in terms of the L_2-norm on C, and he concludes that the method should work satisfactorily for analytic curves C. Numerical tests are announced. An extension of the method due to C. Y. Wang [Thesis, Univ. of Minnesota, 1953] is mentioned.

The last method is one of perturbation of the Lichtenstein-Gershgorin integral equation. The perturbation refers to the kernel K, which is substituted by the kernel K_0 of a region with known mapping function. Numerical experience is not presented; however some aspects of this method, developed by D. Zeitlin, P. C. Rosenbloom and the author are dealt with. *H. Bückner.*

Good, I. J. On the numerical solution of integral equations. Math. Tables Aids Comput. 11 (1957), 82–83.

One of the standard methods of solving linear integral equations with a kernel $K(s,t)$ involves replacing the integral operation by a finite sum in the sense of numerical quadrature. The method leads to the inversion of a matrix M, the elements of which are computed from the values of $K(s,t)$ at certain mesh-points. It is proposed to keep the order of the matrix relatively small and to construct from the inverse matrix an approximation to the inverse kernel by a process of smooth interpolation. The approximation can be used as a good guess for iterative procedures. *H. Bückner* (Schenectady, N.Y.).

Uhlmann, Werner. Fehlerabschätzungen bei Anfangswertaufgaben gewöhnlicher Differentialgleichungssysteme 1. Ordnung. Z. Angew. Math. Mech. 37 (1957), 88–99. (English, French and Russian summaries)

In this paper error estimation formulas are derived for the approximate solution of systems of first order ordinary differential equations when the approximate solutions may be interpreted as a piecewise continuously differentiable function. In particular certain polynomial

approximation formulas are associated with Adam's method and with central difference methods, and deviation functions obtained on substitution into the differential equation system are termed defects. Defect functions are used in the error analysis of arbitrary stepwise procedures (equal steps) for simultaneous first order systems. For a system of two differential equations applications of the general theory and more detailed inequalities are given. Finally numerical examples are given for three problems each involving one first order differential equation. *P. C. Hammer* (Madison, Wis.).

Uhlmann, Werner. Fehlerabschätzungen bei Anfangswertaufgaben einer gewöhnlichen Differentialgleichung höherer Ordnung. Z. Angew. Math. Mech. **37** (1957), 99–111. (English, French and Russian summaries)

In this paper the author discusses errors in numerical solutions of higher order ordinary differential equations along somewhat similar lines to the paper reviewed above. Second order differential equations are treated in additional detail and several numerical examples of these are given. *P. C. Hammer* (Madison, Wis.).

Bertram, G. Fehlerabschätzung für die zweite Randwertaufgabe der ebenen Potentialtheorie. Z. Angew. Math. Mech. **36** (1956), 1–35. (English, French and Russian summaries)

Klamkin, M. S. On a graphical solution of linear differential equations. Amer. Math. Monthly **64** (1957), 428–431.

Lyusternik, L. A.; and Akušskiĭ, I. Ya. On a method of numerical harmonic analysis using a large number of points. Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. **1956**, no. 4(8), 80–85. (Russian)

The authors describe in detail numerical harmonic analysis using a punched card tabulator, a sorter, and a desk calculator. A parallel arrangement on cards permits several problems to be analyzed at the same time. *D. H. Lehmer* (Berkeley, Calif.).

Penndorf, Rudolf; and Goldberg, Bernice. New tables of Mie scattering functions for spherical particles. Air Force Cambridge Research Center, Bedford, Mass., Geophysical Research Papers No. **45** (1956). iii+232 pp.

The functions concerned are

$$a_m(\alpha, n)=(-1)^{m+\frac12}\frac{S_m'(\beta)S_m(\alpha)-nS_m'(\alpha)S_m(\beta)}{S_m'(\beta)\Phi_m(\alpha)-n\Phi_m'(\alpha)S_m(\beta)},$$

$$b_m(\alpha, n)=(-1)^{m+\frac12}\frac{nS_m'(\beta)S_m(\alpha)-S_m'(\alpha)S_m(\beta)}{nS_m'(\beta)\Phi_m(\alpha)-\Phi_m'(\alpha)S_m(\beta)},$$

in which $S_m(x)$ and $C_m(x)$ are the Riccati-Bessel functions $\sqrt{(\frac12\pi x)}J_{m+\frac12}(x)$ and $(-1)^m\sqrt{(\frac12\pi x)}J_{-m-\frac12}(x)$, $\Phi_m(x)=S_m(x)+iC_m(x)$, the primes denote derivatives with respect to the argument α or β, and $\beta=\alpha n$.

The tables give real and imaginary parts of a_m and b_m for $n=1.40$, and for $\alpha=0.1(.1)30$, $m=0(1)m_\alpha$. A few typical values of m_α are

α	0.1	1	2	5	10	20	30
m_α	2	5	7	12	18	31	43.

The reproduction is very bad, and evidently not in accordance with the authors' instructions. The authors claim accuracy to 6 significant figures, and state that they give 7, unrounded, of 10 computed. In fact all 10 appear, though not always completely legibly. The authors cannot, however, escape criticism for not providing better copy for reproduction. *J. C. P. Miller.*

★ **Gloden, A. Table de factorisation des nombres $2N^2+1$ pour $500<N\leq1000$. 2eme éd.** Published by the author, Luxembourg, 1957. 6 pp. (polycopiées)

Complete factorizations of the numbers mentioned in the title are given. *D. H. Lehmer* (Berekley, Calif.).

★ **Gloden, A. Liste des formes linéaires des nombres dont le carré se termine dans le système décimal par une tranche donnée de 4 chiffres. 2e éd.** Chez l'auteur, Luxembourg, 1957. 9 pp. (polycopiées) 40 fr. belges.

Corresponding to each of the 1044 four-digit endings of squares in the decimal system the author lists the arithmetic progressions which characterize those numbers whose squares end in the specified four digits. Thus opposite the entry 6224 one finds $2500n\pm668$. The different moduli involved are 5000, 2500, 1000, 500, 200, and 100. The table is a convenient aid in the solution of many diophantine problems involving squares. *D. H. Lehmer.*

★ **Деканосидзе, Е. Н. [Dekanosidze, E. N.] Таблицы цилиндрических функций от двух переменных. [Tables of cylindrical functions of two variables.]** Izdat. Akad. Nauk SSSR, Moscow, 1956. 495 pp. (3 plates). 50.6 rubles.

The tabulated functions are the "Lommel's functions of two variables" of the standard indexes of Fletcher, Miller and Rosenhead [An index of mathematical tables, McGraw-Hill, New York, 1946; MR **8**, 286] and Lebedev and Fedorova [Spravocnik po matematiceskim tablicam, Izdat. Akad. Nauk SSSR, Moscow, 1956; MR **18**, 828]. The indexes show no substantial prior tabulation of these functions.

The functions tabulated are

$$U_\nu(w, z)=\sum_{m=0}^{\infty}(-1)^m(w/z)^{\nu+2m}J_{\nu+2m}(z)$$

and

$$(*)\qquad V_\nu(w, z)=U_{-\nu+2}(w, z)+\cos(w/2+z^2/2w+\nu\pi/2),$$

where $J_t(z)$ is a Bessel function. They arise in problems of diffraction of electromagnetic waves.

Values of U_1, U_2, V_1, V_2 are tabulated together to 6D for $w=0.5(0.2)1.2(.05)\ 4.0(.1)10.0$ and $z=w(.01)z_L$. Here $z_L=4w^{\frac12}$ for $0.5\leq w\leq6.25$; and $z_L=10$ for $6.25\leq w\leq10$. No differences or other auxiliary material are tabulated. Two plates plot values of $U_\nu(w, z)$ and $V_\nu(w, z)$ for a selected region of (w, z, ν). A third plate shows the (w, z)-region of tabulation. There is a 4-page introduction.

U_ν and V_ν were computed from given infinite series representing U_ν in terms of ordinary Bessel functions of integral orders. Formula (*) was used as the control. Errors are said not to exceed one unit of 10^{-6}, but no further explanation is given. Directions for interpolation indicate that linear, quadratic, or cubic interpolation will be needed, depending on the function and variable considered. For $V_\nu(w, z)$ cubic interpolation with respect to w yields only 4 correct decimals. Recurrence formulas are given by which $U_{\nu+2}(w, z)$ can be obtained from $U_\nu(w, z)$ and $J_\nu(z)$, etc. Other formulas can extend the range to where $w<0$, $z<0$, or $z<w$. There are numerical examples of the use of the formulas, and 11 references.

An acknowledgement states that the tables were computed at the Institute for Exact Mechanics and Computational Engineering (Moscow), but there is no mention of the machine used {BESM?}. The tables appear to be set by hand, and are easily read. The paper is good.

G. E. Forsythe (Stanford, Calif.).

Head, J. W.; and Wilson, W. Proctor. Laguerre functions: Tables and properties. Proc. Inst. Elec. Engrs. C. **103** (1956), 428–440.

This paper gives a considerable collection of formulae and properties of the Laguerre polynomials

$$L_n(x)=\frac{e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^n)$$

and Laguerre functions

$$\lambda_n(x)=e^{-\frac{1}{2}x}L_n(x).$$

These are accompanied by tables as follows: Tables 1–4 give 4D values of $\lambda_n(x)$ for $n=0(1)20$ and

$$x=0(.1)1(.2)3(.5)6(1)14(2)40(5)100;$$

table 5 gives to 5D all zeros of $\lambda_n(x)$ for $n=0(1)20$. There is a figure of a 3-dimensional diagram for $n=0(1)10$, $x\leq 20$.

J. C. P. Miller (Cambridge, England).

Belford, G.; Laslett, L. Jackson; and Snyder, J. N. Table pertaining to solutions of a Hill equation. Math. Tables Aids Comput. **11** (1957), 79–81.

This note describes the range of a table for the characteristic exponent and a function associated with the phase and amplitude for solutions of Hill equations. These equations have the form

$$\frac{d^2Y}{dt^2}+(A+B\cos 2t+C\cos 4t+D\cos 6t)Y=0.$$

Writing solutions of equations in vector-matrix form

$$\mathbf{Y}(t+\pi)=M(t)\mathbf{Y}(t),$$

where $\mathbf{Y}(t)$ has components $y(t)$, $Y'(t)$ then the quantities calculated were $\frac{1}{2}$ trace M which is independent of t and the matrix element $M_{12}(t)$. The quantities were calculated for a variety of values of the parameters A, B, C, and D and t.

P. C. Hammer (Madison, Wis.).

Southard, Thomas H. Approximation and table of the Weierstrass $\wp$ function in the equianharmonic case for real argument. Math. Tables Aids Comput. **11** (1957), 99–100.

The function considered is the Weierstrass function $\wp(u; g_2, g_3)$ with $g_2=0$, $g_3=1$ and real. An approximation to 7 significant figures is given to $\wp(u; 0, 1)$ which is cubic in $y=u^6$ and valid for $0<u\leq w_2 \doteq 1.52995$.

A seven decimal table of the function $\wp(u; 0, 1)-1/u^2$ is also given, with modified second differences, for $u=0(.1).8(.05)1.55$.

J. C. P. Miller.

See also: Brenner, p. 115; Selmer, p. 120; Dingle, p. 133; Fort, p. 151; Crew, Hill and Lavatelli, p. 183; Resnikoff and Lieberman, p. 187; Foster and Rees, p. 188; Vajda, p. 232; Harris, p. 234.

Computing Machines

★ **Stibitz, George R.; and Larrivee, Jules A. Mathematics and computers.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1957. viii+228 pp. $5.00.

One of the authors of this book (G.R.S.) is a pioneer in the field of modern computers and writes with the authority of long experience in their design and many years of thought on their relation to mathematics on the one hand and the outside world on the other. The book is not a textbook in the narrow sense. It has breadth and humor, a wide range of topics and considerable originality. Certainly, no book at all like it has been published before.

The first chapter is entitled "Mathematics, Computers and Problems". It starts from the ground up. Numbers and functions lead to differential equations. The second chapter, "Applied Mathematics and Solutions", tells us that different people, and different mathematicians, mean different things by the word "solution"; it is full of pertinent observations about the "schism in the mathematical family". There follows a discussion of the different kinds of problems computers have to deal with: business, industry, engineering, statistics, science and a brief history of computers, from the abacus, via Babbage, to Aiken, Stibitz, Mauchly, Eckert, and others.

Chapter 5, on numerical analysis, is cursory and little more than an introduction to the concepts of approximate solutions in general and successive approximations in particular. A few pages each are devoted to integration, interpolation, differential equations — ordinary and partial: all distinguished by well chosen examples and illuminating remarks. Chapters 6, 7 and 8 are on Digital Computer Components, Logical Design of Digital Computers, and Analogue Computers and Simulators, respectively. They are of greater interest to mathematicians than such chapters often are. The work concludes with a discussion of random numbers, Monte Carlo methods and games; some notes on computer errors; and a final chapter, Computers at Work.

The novice and the specialist alike will read the book with the pleasure of being amused and at the same time instructed; both will gain insight into the nature of pure and applied mathematics, and of computers, which today form a bridge across the artificial gap separating the two.

W. F. Freiberger (Providence, R.I.).

Stein, A. H. Analysis of closed loop systems. Calc. Automat. y Cibernet. **5** (1956), no. 14, 11–22.

An analogy between analogue and digital computers is studied. The author starts with the example of a simple closed loop control system and sets up its transfer function. He then turns to such special digital computers which receive and deliver a continuous signal, e.g. real time computers for control purposes. He assumes that the computer consists of a modulator of impulses, a digital filter, a demodulator and a carrier of pulses. The input signal is sampled at regular intervals; the samples are quantized and digitalized for computer requirements. Transfer functions of the elements are set up and linked together.

H. Bückner (Schenectady, N.Y.).

★ **Forbes, George F. Digital differential analyzers.** 3rd ed. Pacoima, Calif., 1956. xii+154+8+4+4+3+ix pp. $7.50.

This book is intended for the mathematician and the engineer. It covers the applications of both digital and Bush-type differential analyzers. Some technical information about the elements of the computing machines is given, but the largest part is devoted to set-ups for ordinary differential equations, especially for the generation of the elementary functions and some higher transcendental functions, which satisfy algebraic differential equations. There are also chapters on discontinuous

functions, on differentiation, on accuracy and numerical instability as well as on simultaneous algebraic equations. A trajectory problem is dealt with in detail. It seems that a rather complete survey of what is known for ordinary differential equations and functions of a single variable is given. In addition the author endeavors to apply the differential analyser to problems with more than one independent variable. It is shown how functions of several variables can be calculated from the coefficients of their total differentials; complex functions and especially complex polynomials are included. There is also a discussion of conformal mapping and of applying the differential analyzer to partial differential equations. Author's representation of these subjects is rather involved. To mention one example: On pp. 115–117 he considers complex functions $f(z)=u+iv$, sets up the Cauchy-Riemann equations for a polar coordinate system r, θ, and ends with the equations, $u_x=\sin 2\theta\cdot\phi(\theta)\psi(r)$, $u_y=\sin 2\theta\cdot\phi(\theta)\psi(r)$, where ϕ, ψ are continuous but else arbitrary. It is not mentioned that $f(z)$ has to be linear in z, if these equations hold together with the Cauchy-Riemann equations.

H. Bückner (Schenectady, N.Y.).

Spring, Osc. W.; und Leepin, Peter. Elektronische Rechenmaschinen in Versicherungsbetrieben. Erster Bericht der Kommission zum Studium elektronischer Maschinen in Versicherungsbetrieben. Mitt. Verein. Schweiz. Versich.-Math. **56** (1956), 149–258.

Yavlinskiĭ, N. A. Fast computers and automation of production processes. Èlektričestvo **1956**, no. 9, 7–13. (Russian)

Bruk, I. S. Fast electronic computer $M-2^1$. Èlektričestvo **1956**, no. 9, 14–22. (Russian)

★**McCracken, D. D. Digital computer programming.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1957. vii+253 pp. $7.75.

This book represents an introduction to the field of programming automatic, digital, stored program computers. It fulfills a definite need in that previous publications have dealt largely with specific machines or advanced topics.

The art of programming is demonstrated by means of a mythical computer, TYDAC. This is a decimal, one-address machine, with index registers, a high-speed store of 2000 words and four magnetic tapes.

Among subjects treated are: number systems, decimal point location methods, address modification, loops, flow charts, subroutines, floating-decimal methods, input-output, magnetic tapes, checkout, relative and interpretive programming, double precision arithmetic.

The emphasis throughout is on problem solving and programming techniques. There is little mention of computer logic on which there is already an excellent book in existence [R. K. Richards, Arithmetic operations in digital computers, Van Nostrand, New York, 1955; MR 17, 672]. Nor does it deal with the programming problems posed by advanced chapters of numerical analysis, on which a book is still to be written. The examples discussed are fairly elementary and the stress is on logical rather than mathematical problems. The chapter on interpretive programming whetted one's appetite for more details on the anatomy of interpreters and compilers — a subject very hard to come by in published work.

The book is well produced, highly readable and, particularly as a "first" of its kind, greatly welcome. It is suitable as a text book for computer courses on all but the more advanced levels.

W. Freiberger.

Messerle, H. K. Utilisation d'un analyseur différentiel pour résoudre des problèmes hydrauliques posés par des aménagements hydroélectriques. Houille Blanche **11** (1956), 813–836. (English and French)

This is a report on the application of mechanical differential analyzers with integrators, adding units, input and output tables for finding the best combination of hydraulic design parameters of a complex hydro-electric system, involving water storages and tunnels. The calculations require a basic integration of the type

$$\int_0^t [Q_i(t)-Q_0(H,t)]\cdot dt=S(H)-S_0 \quad (t=\text{time}),$$

where H is the water level of a reservoir, $S(H)$ the water stored, Q_i the inflow rate and Q_0 the outflow rate. It was found, that the differential analyzer is very useful in order to carry out the calculations of a large number of flood-routings. Results are presented for the Tooma-Tumut Diversion, a part of the Snowy Mountains Hydro Development in New South Wales.

H. Bückner.

Crew, J. E.; Hill, R. D.; and Lavatelli, L. S. Monte Carlo calculation of single pion production by pions. Phys. Rev. (2) **106** (1957), 1051–1056.

The high speed digital computer ILLIAC at the Digital Computer Laboratory of the University of Illinois was employed in Monte Carlo type calculations. These calculations involving the successful application of a random number generator concerned single pion production in a pion-nucleon collision using a 3/2, 3/2 isobaric nucleon model. No details concerning the mathematical techniques used in the numerical solution were given.

Ruth M. Davis (Washington, D.C.).

Kulikov, D. K. Application of computing machines to the calculation of heliocentric coordinates of the major planets. Byull. Inst. Teoret. Astr. **6** (1955), 166–191. (Russian)

Bohan, N. A. Integration of the equations of motion of small planets using computing machines. Byull. Inst. Teoret. Astr. **6** (1955), 162–165 (1 plate). (Russian)

Kulikov, D. K. On subtabulation of tables on computing machines. Byull. Inst. Teoret. Astr. **6** (1955), 192–201. (Russian)

See also: Lehmer, p. 121; Lehmer, p. 123.

PROBABILITY

Gheorghiu, Şerban. Quelques problèmes concernant la division d'un segment par des points pris au hasard. Rev. Math. Pures Appl. 1 (1956), no. 1, 99–124.

A translation from the Romanian of the article reviewed in MR 17, 1095.

Seal, K. C. On a characterization of gamma distributions. Calcutta Statist. Assoc. Bull. 7 (1957), 60–72.

Let $X_1, X_2, \cdots, X_n$ be n identically and independently distributed random variables with finite second moment. R. G. Laha [Ann. Math. Statist. 25 (1954), 784–787; MR 16, 269] has shown that the stochastic independence of the ratio $(a_1X_1+a_2X_2+\cdots+a_nX_n)/(X_1+X_2+\cdots+X_n)$ and of the sum $(X_1+X_2+\cdots+X_n)$ implies that the common distribution of the $X_1, \cdots, X_n$ is a gamma distribution. The author first shows that for $n>2$ the assumption that the $X_1, \cdots, X_n$ are identically distributed is essential. He then derives several results. As a typical example we mention the following theorem: Let $X_1, \cdots, X_n$ be n mutually independent (but not identically distributed) proper random variables. The random variables $X_1, \cdots, X_n$ have all gamma distributions with the same scale parameter if, and only if, the ratio $\sum_1^n a_{kt}X_t/\sum_1^n X_t$ is independent of $\sum_1^n X_t$ for each $k=2, 3, \cdots, n$. The author's results are closely related to work by R. G. Laha [loc. cit.] and the reviewer [ibid. 26 (1955), 319–324; Proc. 3rd Berkeley Symposium, Math. Statist. and Probability, 1954–1955, vol. 2, Univ. of California Press, 1956, pp. 195–214; MR 16, 1034; 18, 942]. *E. Lukacs.*

Lukacs, Eugène. Sur une transformation des fonctions caractéristiques. C. R. Acad. Sci. Paris 244 (1957), 2467–2468.

The author proves that if $f(y)$ is an arbitrary characteristic function then

$$-\int_0^t\int_0^u f(y)dydu$$

is the second characteristic of an infinitely divisible law with a finite second moment. The proof uses Kolmogorov's [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 15 (1932), 805–808] canonical representation of the second characteristic $\psi(t)=\log f(t)$ of an infinitely divisible distribution with finite second moment. As applications, he shows that the function $f(t)-1$ is the second characteristic of an infinitely divisible law [see de Finetti, ibid. 12 (1930), 278–282] and that it is possible to decompose the Cauchy law into two factors which are not stable laws [see Dugué, C. R. Acad. Sci. Paris 213 (1941), 718–719; MR 5, 124]. *H. P. Edmundson* (Santa Monica, Calif.).

Girault, M. Produit de fonctions caractéristiques et indépendance de variables aléatoires. Publ. Inst. Statist. Univ. Paris 5 (1956), 29–31.

The author elaborates on the fact that the distributions of random variables X, Y and $X+Y$ do not determine the joint distribution of (X, Y). *S. C. Moy.*

Fisz, M. Refinement of a probability limit theorem and its application to Bessel functions. Acta Math. Acad. Sci. Hungar. 6 (1955), 199–202. (Russian summary)

The difference between two independent Poisson variables is asymptotically normal. The author finds an asymptotic expansion for the probability of this difference using the general error estimate of Esseen. With the help of this expansion he obtains an asymptotic formula for Bessel functions of order k with pure imaginary argument. *M. D. Donsker* (Minneapolis, Minn.).

Stanojević, Časlav V. On a theorem of K. L. Chung. Bull. Soc. Math. Phys. Serbie 8 (1956), 59–60. (Serbo-Croatian. English summary)

Let X_n be a sequence of independent random variables, with $M(X_n)=0$, $M(|X_n|^{2r})<\infty$ for some $r\geqq 1$; and let $n\varepsilon^{2r}\uparrow\infty$, $\varepsilon_n\downarrow 0$. If $\sum_n n^{-r-1}\varepsilon_n^{-2r}M(|X_n|^{2r})<\infty$, then $P\{\lim_{n\to\infty}(n\varepsilon_n)^{-1}S_n=0\}=1$. The proof follows that of the reviewer where $\varepsilon_n=1$. [Proc. 2nd Berkeley Symposium on Math. Statist. and Probability, 1950, Univ. of California Press, 1951, pp. 341–352; MR 13, 567]. In this case the result is due to Kolmogorov if $r=1$ [C. R. Acad. Sci. Paris 191 (1930), 910–912]; to H. D. Brunk if $n\geqq 1$ is an integer [Duke Math. J. 15 (1948), 181–195; MR 9, 450]. *K. L. Chung* (Syracuse, N.Y.).

Kac, M. A class of limit theorems. Trans. Amer. Math. Soc. 84 (1957), 459–471.

Let $X_1, X_2, \cdots$ be independent random variables having the same even density function $p(x)$ whose characteristic function $\varphi(\xi)$ is absolutely integrable in $(-\infty, \infty)$ and such that

$$\varphi(\xi)\approx 1-|\xi|^\gamma,\quad \xi\to 0,\quad 1\leqq\gamma\leqq 2.$$

Denote by $P_n(j;\Omega)$ the probability that exactly j among the first n partial sums $s_k=X_1+\cdots+X_k$ fall within Ω, a bounded measurable set.

The author proves that the limits

$$\lim_{n\to\infty} n^{1-1/\gamma}P_n(j;\Omega)\quad (1<\gamma\leqq 2)$$
$$\lim_{n\to\infty} (\log n)P_n(j;\Omega)\quad (\gamma=1)$$

exist and are given in terms of the eigenvalues and eigenfunctions of the integral equation

$$\int_\Omega L(x-y)\psi(y)dy=\lambda\psi(x),$$

where

$$L(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-\cos x\xi}{1-\varphi(\xi)}\,\varphi(\xi)d\xi.$$

The function $-L(x)$ is the "finite" part of the generating function of the transition probabilities for the Markov chain s_k — the corresponding limit theorems for the "infinite" part having been obtained by the author and reviewer previously [same Trans. 84 (1957), 444–458; MR 18, 832].

The theory is simpler, and analogous results obtain, when the X_i are lattice-valued. The more difficult extension to continuous parameter processes is directly related to certain problems in potential theory, on which the author promises a subsequent publication. *D. A. Darling* (Ann Arbor, Mich.).

Dobrushin, R. Central limit theorem for non-stationary Markov chains. I. Teor. Veroyatnost. i Primenen. 1 (1956), 72–89. (Russian. English summary)

This paper consists of the introduction to a dissertation and includes, in addition to statements of theorems, a brief outline of some of the proofs. The principal theorems are refinements upon those announced previously by the

author [Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 5–8; MR **17**, 48] and will be indicated below with the notations of the review just cited. For any transition probability function $P(x, A)$, instead of the measure of ergodicity ρ the author now uses the "ergodic coefficient"

$$\alpha=\alpha(P)=1-\sup|P(x, A)-P(y, A)|,$$

where the supremum is taken for all x, y, and A. Replacing ρ by α in theorems 1) and 2) of the review cited we get Theorems 1 and 2 of the present paper, which are somewhat stronger because $\alpha\geqq\rho$. Moreover, in them the condition involving α_n is best possible. Instead of 3) of the review the author states the following more general result (Theorem 8): under condition (*) of the review if $\alpha_n n^{1/3}\to\infty$, if $F_i^{(n)}(t)$ denotes the cumulative distribution function of $\xi_{in}-\mathscr{E}[\xi_{in}]$, and if, for any $r>0$,

$$\lim_{n\to\infty}\frac{1}{n\alpha_n^2}\sum_{i=1}^{n}\int_{t^2\geq r^2 n\alpha_n^2} t^2 dF_i^{(n)}(t)=0,$$

then ζ_n' is asymptotically normally distributed with mean 0 and variance 1. *H. P. Mulholland* (Birmingham).

Wolfowitz, J. On stochastic approximation methods. Ann. Math. Statist. **27** (1956), 1151–1156.

In a recent paper [Proc. 3rd Berkeley Symposium Math. Statist. and Probability, 1954–1955, v. 1, Univ. of California Press, 1956, pp. 39–55; MR **18**, 946] the reviewer proved a theorem on stochastic approximation which implied all previous results on convergence in probability and in the mean of stochastic approximation methods. The purpose of the present note is to provide a somewhat shorter and possibly more perspicuous proof of the above theorem. Some of the devices employed by the reviewer are retained, but a more concise notation is used and convergence in probability is proved before establishing convergence in the mean (thus reversing the order in the above mentioned paper). *A. Dvoretzky.*

Lévy, Paul. Sur quelques problèmes de la théorie des liaisons stochastiques. C. R. Acad. Sci. Paris **244** (1957), 1313–1316.

Statulyavičus, V. A. Asymptotic expansion for inhomogeneous Markov chains. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 206. (Russian)

The author continues his investigation of an inhomogeneous Markov chain with s states [same Dokl. **107** (1956), 516–519; MR **19**, 71]. Let e_γ denote the initial state and let the random variable $\zeta_n^{(\alpha)}$ denote the number of visits to state e_α after the first n steps. Under the assumptions (A) $p_{\alpha\beta}^{(l)}\geqq\lambda p_{\alpha\beta}^{(k)}$ for any α, β, k, and l where $\lambda>0$, (B) the set of states of the chain form one essential class, and (C) the chain's rank r equals the number of states s, he derives an asymptotic expression for the probability $P_\gamma(m)$ that the random vector $\zeta_n=(\zeta_n^{(1)}, \cdots, \zeta_n^{(s)})$ assumes the vector value $m=(m_1, \cdots, m_s)$.

H. P. Edmundson (Santa Monica, California).

Elliott, Joanne; and Feller, William. Stochastic processes connected with harmonic functions. Trans. Amer. Math. Soc. **82** (1956), 392–420.

The considerations of this paper center about the absorbing barrier process on $(-a, a)$ connected with the Cauchy process. This process was first studied by M. Kac [Proc. 2nd Berkeley Symposium Math. Statist. Probability 1950, Univ. of California Press, 1951, pp. 189–215; MR **13**, 568] and later by J. Elliott [Trans. Amer. Math. Soc. **76** (1954), 300–331; MR **15**, 715; see also the paper reviewed below]. The infinitesimal generator for this absorbing barrier process is

$$\Omega_a F=\pi^{-1}P\int_{-a}^{a}\frac{F'(y)}{y-x}\,dy,\quad F(\pm a)=0,$$

the integral being taken in the sense of a Cauchy principal value. Proceeding from this process the authors arrive at the Cauchy process on $(-\infty, \infty)$ by means of an iterative procedure and as a by-product they obtain a characterization of the infinitesimal generator for the Cauchy process which is of the form Ω_∞. Next, the absorbing barrier process is combined with a return process defined as follows: When the path leaves the interval $(-a, a)$ for the first time, jumping say to the point z with $|z|>a$, it returns to a point $Y\in(-a, a)$ with given probability distribution $\Pr\{Y\in S\}=\tau(z)p(z, S)$, where $0\leqq\tau(z)\leqq 1$ and $p(z, (-a, a))=1$. The process then continues as before from the point Y. The transition probabilities of this process are computed and the infinitesimal generator of the associated semi-group is characterized. It should be remarked that because of the discontinuous character of the Cauchy process, the infinitesimal generator is no longer of the form of Ω_a. The last section of the paper is devoted to still another connection of the absorbing barrier process on $(-a, a)$ with the Cauchy process on $(-\infty, \infty)$. Applying the latter process to periodic functions of period $2a$ or to such functions with

$$f'(-a)=0=f'(a)$$

one obtains processes which when limited to $(-a, a)$ can be characterized as absorbing barrier processes ion $(-a, a)$ combined with certain return processes. In this case the functions defined by the semi-group solutions are harmonic in (x, t). The arguments are for the most part analytical and depend to a large extent on results from the above mentioned Elliott papers. *R. S. Phillips.*

Elliott, Joanne. On an integrodifferential operator of the Cauchy type. Proc. Amer. Math. Soc. **7** (1956), 616–626.

In previous papers [Trans. Amer. Math. Soc. **76** (1954), 300–331; MR **15**, 715; see also the paper reviewed above] the author has studied functional equations associated with a Cauchy process for a finite interval. One of these is the resolvent equation

$$\text{(1)}\qquad \lambda(Fx)-\pi^{-1}P\int_{-a}^{a}F'(t)(t-x)^{-1}dt=f(x),\ \lambda>0,$$

which for $f(x)\in C[-a, a]$ has a solution of the form

$$\text{(2)}\qquad F(x)=\int_{-a}^{a}\Gamma_a(x, y; \lambda)f(y)dy$$

with a non-negative symmetric kernel Γ_a. In the present note $f(x)$ is said to belong to $U[-a, a]$ if

$$(a^2-x^2)f(x)\in C[-a, a].$$

If $f\in U$, $F(x)\in C[-a, a]$ when $F(x)$ is defined by (2). In general this $F(x)$ will not satisfy (1) but a more complicated equation in which the second term on the left is replaced by

$$\pi^{-1}(a^2-x^2)^{-1}\{P\int_{-a}^{a}F'(t)(a^2-t^2)(t-x)^{-1}dt-\int_{-a}^{a}F(t)dt\}.$$

The solution of this equation for $f\in U[-a, a]$ is determined and proved to be unique in $C[-a, a]$. A number of intermediary results were used in the second paper quoted above. *E. Hille* (New Haven, Conn.)

Fürst, Dario. La rovina dei giocatori nel caso di riserva limitata. Giorn. Ist. Ital. Attuari **19** (1956), 63–83.

The paper studies not necessarily symmetric random walk with one absorbing and one reflecting barrier, with special reference to insurance reserves. A variant in which the steps occur according to a Poisson process is also treated. *L. J. Savage* (Chicago, Ill.).

See also: Le Cam, p. 128; Kosma'k, p. 134; Ha'jek, p. 134; Bartlett, p. 233; Cohen, p. 235.

STATISTICS

Ikenberry, Ernest. Characteristics and convergence of Gram-Charlier series. Statistica, Bologna **17** (1957), 3–10.

The Type A Gram-Charlier series is considered:

$$(1) \qquad f_R(x)=\sum_{r=0}^{R-1} c_r H_r(t)g(t),$$

where $t=(x-\bar{x})/s$, $\bar{x}$ and s are computed from a sample of size n, $g(t)$ is the Gaussian density function, and

$$H_r(t)g(t)=(-1)^r \frac{d^r}{dt^r}\, g(t).$$

It is shown that, if the c_r are computed as (for g based on sample data)

$$c_r=(1/r!)\sum_j g(x_j)H_r(t_j),$$

the sum of a finite number of terms may give negative computed frequencies in (1) (particularly near the tails), modes contrary to the data and accentuated maximas. Values of R (in terms of n, s and the range, w) are given for which these results are most likely to occur. In addition (1) does not converge as $R\to\infty$.

If the c_r are computed in terms of a known $f(x)$, the probability of the convergence of (1) depends upon the value of s. For Gaussian $f(x)$, (1) converges if $s^2>\sigma^2/2$; hence the probability of convergence is simply

$$P\{x_{n-1}^2>(n-1)/2\}=\alpha.$$

Values of α are given for $n-1=1(1)30$.
R. L. Anderson (Raleigh, N.C.).

Kullback, S.; and Rosenblatt, H. M. On the analysis of multiple regression in k categories. Biometrika **44** (1957), 67–83.

Let y_{ji} be the ith observation in the jth category of a random variable with mean value $E(y_{ji})=\sum_{k=1}^{p} \beta_{jk}x_{ijk}$ depending on p known variables (the x's) and unknown regression coefficients (the β's). Let the y_{ji} be independent and normally distributed with unknown variance σ^2. The authors work out in detail the usual tests of the hypotheses that (i) $\beta_{jk}=\beta_{.k}$ (all j, k), and (ii) $\beta_{jk}=\beta_{.k}$ (all j, some k). The results are applied to a problem studied by Carter [Biometrika **36** (1949), 26–46; MR **11**, 673]. A numerical example is included. The justification for the test of the linear hypothesis is based on information theory ideas developed by Kullback [Ann. Math. Statist. **23** (1952), 88–102; MR **13**, 854]. The property that the test has of being uniformly most powerful amongst invariant tests is surely a much stronger recommendation for its acceptance. *D. V. Lindley.*

Brown, R. L. Bivariate structural relation. Biometrika **44** (1957), 84–96.

The variables X, Y connected by $Y=\alpha_0+\alpha_1 X$ are observed as $x=X+\varepsilon$, $y=Y+\theta$, where ε and θ are independent $N(0, 1)$ random variables. It is remarked that in a random sample the sum of the squares of the perpendicular distances from the observed points to the line is distributed as χ^2, a distribution not involving the parameters. The usual type of inversion produces a joint confidence set for α_0 and α_1. An acceptance conic is derived therefrom, which contains all lines at a prescribed confidence level. An attempt is made to extend the idea to the case where the standard deviations of ε and θ are unknowns and to the curvilinear case $Y=\sum \alpha_j\phi_j(X)$. The paper contains several remarks of a quasi-philosophical character which are not intelligible to the reviewer. No clear distinction is drawn between parameters and random variables. *D. V. Lindley.*

Darwin, J. H. The difference between consecutive members of a series of random variables arranged in order of size. Biometrika **44** (1957), 211–218.

This paper considers the distribution of the difference between the mth and $(m+1)$th largest values in a random sample from a known distribution. First- and second-order approximations are obtained by the method of steepest descent. It is shown that these approximations are rather good even for small samples for the normal, exponential, and rectangular distributions. Also renormalization of the first-order approximation is considered; from the tables presented, this seems inferior to the second-order approximation, and in the tails may even be worse then the first-order approximation. *H. Rubin.*

Matusita, Kameo. Decision rule, based on the distance, for the classification problem. Ann. Inst. Statist. Math., Tokyo **8** (1956), 67–77.

All the distributions considered are assumed to assign all the probability to k given points, $b_1, \cdots, b_k$. If F is a distribution assigning probability p_i to the point b_i, while the distribution G assigns probability q_i to the point b_i, then the distance $\|F-G\|$ between F and G is defined as

$$\sum_{i=1}^{k} (p_i^{\frac{1}{2}}-q_i^{\frac{1}{2}})^2.$$

Suppose we take n observations on a chance variable X, m observations on a chance variable Y, and L observations on a chance variable Z, all observations independent. S_n, S_m', and S_L'' denote the empirical distributions based on the observations on X, Y, and Z respectively. The distributions of X, Y, and Z are unknown, but Z is known to have either the same distribution as X or the same distributions as Y, and our problem is to decide which. The author introduces the following decision rule: When $\|S_n-S_L''\|\leq\|S_m'-S_L''\|$, decide Z has the same distribution as X, otherwise, decide Z has the same distribution as Y. Denoting the distribution of X by F and the distribution of Y by G, the author develops a lower bound for the probability of a correct decision, this bound being a function of $\|F-G\|$.

An analogous rule is defined for cases where the distributions of X and Y are of known functional form, but with certain parameters unknown. *L. Weiss.*

Bazarov, I. P. Equations with variational derivatives and distribution functions for systems with complicated interaction. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 38–41. (Russian)

★ **Resnikoff, George J.; and Lieberman, Gerald J. Tables of the non-central t-distribution: density function, cumulative distribution function and percentage points.** Stanford studies in mathematics and statistics, I. Stanford University Press, Stanford, California, 1957. ix+389 pp. $12.50.

Tables are given for the probability density, probability integral and percentage points of $t=(z+\delta)/(w^{\frac{1}{2}})$. Here z and fw denote the independent unit normal and χ^2 with f degrees of freedom, respectively, and $\delta=(f+1)^{\frac{1}{2}}K_p$, where K_p is such that $\Pr(z>K_p)=1-p$. The argument used is $t/(f^{\frac{1}{2}})$, which for the probability density and the probability integral runs in steps of 0.05. In all tables $f=2(1)24(5)49$ and $p=0.2500$, 0.1500, 0.1000, 0.0650, 0.0400, 0.0250, 0.0100, 0.0040, 0.0025, 0.0010. A table of δ is given for the same values of f and p. Percentage points are given for the percentages 0.5, 1, 5, 10, 25, 50, 75, 90, 95, 99, and 99.5. About 20 pages of the introduction are devoted to numerical illustrations of various applications, e.g., a sequential t-test and acceptance sampling by variables. *D. M. Sandelius* (Göteborg).

Medgyessy, P. Anwendungsmöglichkeiten der Analyse der Wahrscheinlichkeitsdichtefunktionen bei der Auswertung von Messungsergebnissen. Z. Angew. Math. Mech. **37** (1957), 128–139. (English, French and Russian summaries)

"Of certain measurements, e.g., in spectroscopy, it is merely known that they belong to a graph which is a superposition or "mixture" of so-called stable, e.g., normal, frequency functions, while the values of the parameters of these functions, which are the physical quantities to be found, are unknown. The problem of the so-called "analysis of mixtures", i.e., the problem of determining the unknown parameter values, is solved exactly by probabilistic methods. Approximation methods are also discussed, mainly for the case of a mixture of normal or Cauchy frequency functions, respectively." (Author's summary.) *D. M. Sandelius* (Göteborg).

Roy, K. P. A note on the asymptotic distribution of likelihood ratio. Calcutta Statist. Assoc. Bull. **7** (1957), 73–77.

Let X be a random variable whose density function $f(x;\theta)$, $\theta=(\theta_1,\theta_2,\cdots,\theta_k)$ satisfies the two following conditions. (A)

$$(1)\qquad \int_{-\infty}^{+\infty}\frac{\partial \log f}{\partial\theta_i}f(x;\theta)dx=0 \text{ for all } \theta_i\in I,$$

where I is a non-degenerate interval, and

$$(2)\qquad \int_{-\infty}^{+\infty}\left[\frac{\partial^2\log f}{\partial\theta_i\partial\theta_j}\right]_{\theta=\theta'}f(x;\theta)dx=-K_{ij}(\theta,\theta'),$$

where $\theta,\theta'\in I$, and $K_{ij}(\theta,\theta')$ and

$$C_{ij}(\theta')=\left[-\sum_{i=1}^{n}\frac{\partial^2\log f(x_i;\theta)}{\partial\theta_i\partial\theta_j}\right]_{\theta=\theta'}$$

are uniformly continuous in θ, θ'. The author presents a reasonably self-contained derivation of the limiting distribution of $-2\log\lambda$, where λ is the likelihood ratio statistic, under the condition that the hypothesis tested is true. The paper discusses the two cases of simple and composite null hypotheses.

F. C. Andrews (Eugene, Ore.).

David, F. N.; and Johnson, N. L. Reciprocal Bernoulli and Poisson variables. An. Fac. Ci. Porto **37** (1953), 200–203.

Approximate formulae are given for the moments of a random variable which is the reciprocal of a binomial variable which has been truncated to omit the zero value. *M. Dwass* (Evanston, Ill.).

van Elteren, Ph.; and van Peype, W. F. Some rank correlation methods. Statistica, Neerlandica **10** (1956), 177–195. (Dutch. English summary)

The authors explain and examine certain methods of rank correlation. The problems arise in connection with psychological testing of several subjects in extrasensory perception or prediction of marked cards. The authors discuss first the method of m rankings of n given objects, by the elementary scheme of column totals in the m by n matrix. They then explain Kendall's method of paired comparisons, and illustrate measures of consistency by a simple example. When n exceeds 7 the practical details of experiment become overwhelming, and recourse must be had to incomplete block designs. Such a design for $m=n=13$, and comparing only 4 at a time, is illustrated. The authors propose 3 desirable properties of an impartial incomplete block design: 1. In each ranking the same number of objects shall be arranged; 2. Each object shall be examined the same number of times; 3. Each pair of objects shall occur equally often in the individual rankings. An experiment with 200 observed sets of listings with common pattern was studied in comparison with theoretically unbiased data, by the device of using random members as proposed by M. G. Kendall and B. Babington Smith [Tables of random sampling numbers, Cambridge, 1939]. The numerical results are matched with a Pearson Beta-distribution and the comparison exhibited in a graph. *A. A. Bennett* (Providence, R.I.).

Kesten, H.; and Runnenburg, J. Th. Some remarks on the calculation of the expectation and the spread of the number of inconsistencies in a fixed rank correlation scheme. Statistica, Neerlandica **10** (1956), 197–204. (Dutch. English summary)

These authors compute the numerical values of the variance of the mean and derivation respectively of the sum of squares of row totals in the last-discussed pattern of the previous paper. The details of algebraic computation are fully exhibited, making the method applicable generally to analogous patterns. *A. A. Bennett.*

Schäffer, K.-A. Der Likelihood-Anpassungstest. Mitteilungsbl. Math. Statist. **9** (1957), 27–54.

Der Autor entwickelt Formeln fuer die Momente des zum Pruefen einer einfachen multinomialen Verteilung benutzten Likelihood Anpassungsmasses L, und basiert darauf eine Naeherungsmethode zur Berechnung der kritischen Werte von L, die in numerischen Beispielen besser zu sein scheint als die ueblichen Methoden. Er vergleicht an Hand von Zahlenbeispielen die Trennschaerfe dieses Tests mit dem des χ^2-Anpassungstests und kommt zu dem Schluss: "Danach darf der Likelihood-Anpassungstest gegenueber dem χ^2-Verfahren als 'regelmaessig besser' bezeichnet werden." Es scheint dem Referenten, dass sich durch Anwendung derselben Methode auf eine andere Klasse von Punkten das gegenteilige Resultat zeigen laesst.

J. L. Hodges, Jr. (Berkeley, Calif.).

Hogg, Robert V.; and Craig, Allen T. Sufficient statistics in elementary distribution theory. Sankhyā 17 (1956), 209–216.

A theorem of Neyman [Statist. Res. Mem., London 2 (1938), 58–59] and Basu [Sanskhyā 15 (1955), 377–380; MR 17, 640] is used to prove independence of certain statistics in a number of cases concerning normal, rectangular and exponential distributions. *E. L. Lehmann.*

Čžan, Li-Cyan'. On the precise distribution of A. N. Kolmogoroff and its asymptotic analysis. Acta Math. Sinica 6 (1956), 55–81. (Chinese. Russian summary)

Let $x_1, x_2, \cdots, x_n$ be the results of n independent trials of the random variable X with continuous distribution function $F(x)$. Let $F_n(x)$ denote the empirical distribution of choices, i.e., $nF_n(x)$ is the number of the x_k which are less then x. Let us set

$$D_n = \sup_{-\infty<x<+\infty} |(F_n(x)-F(x)|.$$

A. N. Kolmogorov [Sulla determinatione empirica di una legge di distributione. Giorn. Istituto Ital. Attuari, 4 (1933) 83–91] established the following well-known theorem:

$$\Phi_n(z)=P\left\{D_n<\frac{z}{\sqrt{n}}\right\} \underset{n\to\infty}{\to} \begin{cases} 0, & \text{for } z\leqq 0, \\ \sum_{k=-\infty}^{\infty}(-1)^k e^{-2k^2z^2}, & \text{for } z>0. \end{cases}$$

The present article independently establishes by a direct method the exact distribution $\Phi_n(z\sqrt{n})$ of the variable D_n. *From the author's summary.*

★**Fortunati, Paolo. Rapporto di concentrazione, valori medi e schemi teorici di distribuzione massimante e minimante della variabilità.** Scritti matematici in onore di Filippo Sibirani, pp. 109–122. Cesare Zuffi, Bologna, 1957.

Grenander, Ulf. On the theory of mortality measurement. I. Skand. Aktuarietidskr. 39 (1956), 70–96.

The properties of maximum likelihood estimates of the Makeham parameters are studied. Then, various methods of estimating Makeham mortality are discussed and expressions are given for the asymptotic distributions of these estimates. *P. Johansen* (Copenhagen).

Barton, D. E. The modality of Neyman's contagious distribution of type A. Trabajos Estadist. 8 (1957), 13–22. (Spanish summary)

The author studies discrete frequency distributions with an observed departure from a Poisson law due to a clustering or contagious effect [see Irwin, Suppl. Roy. Statist. Soc. 7 (1941), 101–107; Feller, Ann. Math. Statist. 14 (1943), 389–400; MR 5, 209; and Anscombe, Biometrika 37 (1950), 358–382; MR 12, 510]. Fitting Neyman's contagious distribution of type A, which tends to be multimodal, requires considerable computational labor. The author treats the question of whether the law is multimodal, sesquimodal or unimodal on the basis of the two parameters of the type A distribution. He shows that the distribution function can be considered as the product of a Poisson distribution with parameter λ and a modifying polynomial dependent on I, the index of clumping [see David and Moore, Ann. Botany (N.S.) 18 (1954), 47–53]. He decomposes the (I, log λ) plane into regions representing different modalities. *H. P. Edmundson.*

★**Olds, Edwin G.; and Severo, Norman C. A comparison of tests on the mean of a logarithmico-normal distribution with known variance.** WADC Tech. Note 55-249. Wright Air Devel. Center, Wright-Patterson Air Force Base, Ohio, 1955. vi+58 pp.

[A shortened version has appeared in Ann. Math. Statist. 27 (1956), 670–686; MR 18, 426.] The present investigation is concerned with the application of the logarithmic transformation to the problem of testing an hypothesis on the mean of a logarithmico-normal variate with known variance. An experimenter can fail to recognize the need for a transformation and simply proceed to apply normal theory tests to the original data; or he can properly transform the data and then apply a normal theory test to a parameter of the transformed scale. Each of these testing procedures is investigated in detail. Finally a third test procedure is developed by using the Neyman-Pearson Lemma for testing simple hypotheses. A comparison of these tests is then made by means of their operating characteristics and some asymptotic properties obtained. It is found that the three procedures give quite different results unless the mean under the null hypothesis is large relative to the standard deviation. (From the authors' introduction.) *H. A. David.*

Foster, F. G.; and Rees, D. H. Upper percentage points of the generalized beta distribution. I. Biometrika 44 (1957), 237–247.

A number of statistical tests in multivariate analysis are based on certain functions of the roots of the determinantal equation $|\nu_2B-\theta(\nu_1A+\nu_2B)|=0$, where A and B are independently distributed, each having a Wishart distribution with ν_1 and ν_2 degrees of freedom and same covariance matrix. In this paper, the test based on the largest root is considered and upper percentage points are tabulated. The table gives the values of x to $4D$ for which

$$\int_0^x d\theta_2 \int_0^{\theta_2} (\theta_1\theta_2)^{(\nu_1-3)/2}[(1-\theta_1)(1-\theta_2)]^{(\nu_2-3)/2}(\theta_2-\theta_1)d\theta_1=P,$$

for $P=.80(.05).95$, $.99$, $\nu_1=5(2)41(10)121$, 161, $\nu_2=2,3(2)21$. This extends the tables of Nanda [J. Indian Soc. Agric. Statist. 3 (1951), 175–177; MR 13, 478] and Pillai [Biometrika 43 (1956), 122–127; MR 17, 983]. Two examples on testing the equality of mean vectors and testing of regression coefficients are carried out. *I. Olkin* (East Lansing, Mich.).

Krishna Iyer, P. V.; and Singh, B. N. On certain probability distributions arising from a sequence of observations and their applications. J. Indian Soc. Agric. Statist. 7 (1955), 127–168.

New test statistics based on rank order are proposed and discussed for testing (1) whether a sequence of observations may be regarded as a random sample and (2) whether two or more independent random samples come from the same population. Given a sequence of random variables, $Y_1, Y_2, \cdots, Y_n$, the following test statistics are defined:

$$W_r=\sum_{k=1}^{n-r+1}\left\{\sum_{0\leq i<j\leq r-1}\sum \operatorname{pos}(Y_{k+j}-Y_{k+i})\right\},$$

$$T_r=\sum_{\substack{1\leq i<j\leq n \\ j-i\leq r-1}}\sum \operatorname{pos}(Y_j-Y_i)$$

for $r=2, \cdots, n$, and where pos $x=1$ or 0 according as $x>0$ or $x\leqq 0$. Analogous statistics are also considered with j and i interchanged within the above sums, and with pos x in the sums replaced by sgn $|x|$. T_r is the number of

positive ordered differences between Y's with subscripts differing by $r-1$ or less; W_r is the sum of T_r's over the $n-r+1$ blocks of contiguous Y's. $T_n=W_n$ is the Mann-Whitney form of the test statistic for Wilcoxon's unpaired two sample test, provided that the Y's take values 1 and 2 and are ordered as the actual observations. Also, $T_n=W_n$ is essentially Kendall's τ (no-ties case) if the Y_i's are ordered by the values of associated observations, X_i. $T_2=W_2$ is the test statistic for a well known test of randomness based on the number of positive differences $Y_{i+1}-Y_i$.

Low order cumulants for the proposed statistics are obtained under the null hypothesis of "free sampling" (i.e., the Y's independent and identically distributed, possibly discrete), and under the null hypothesis of "non-free sampling" (i.e., the sequence of Y's takes on with equal probability each permutation of a fixed set of numbers.) The first of these hypotheses relates to testing randomness, and the second to the several sample problem, where the Y's are sample numbers ordered as the observations. Proofs are indicated for asymptotic normality under the null hypothesis and for consistency.

The power and Pitman asymptotic efficiency of the proposed tests are discussed, and tables of power values and of coefficients of variation are presented, mostly in what is asserted to be the two sample context. On the basis of the tables it is concluded that some of the proposed tests in the two sample case, with normal alternatives, are much more powerful than Wilcoxon's test, and even somewhat more powerful (in a qualified but unclear sense) than the t test! {The meaning of the tables is not clear to me, but they do not seem pertinent to the usual two sample problem.} *W. Kruskal.*

Kiefer, J.; and Wolfowitz, J. Consistency of the maximum likelihood estimator in the presence of infinitely many incidental parameters. Ann. Math. Statist. **27** (1956), 887–906.

Let $\{X_{ij}\}$ $(i=1, \cdots, n; j=1, \cdots, k)$ be random variables such that $X_{i1}, \cdots, X_{ik}$ have a frequency function $f(x|\theta, \alpha_i)$ depending on the structural parameter θ (the same for all i) and the incidental parameter α_i. It is assumed that the X_{ij} are independent for any given θ, $\alpha_1, \cdots, \alpha_n$ and that the α_i are independent random variables with a common (unknown) distribution function G_0. The problem of consistently estimating θ as $n\to\infty$ is studied. Following in essence Wald [same Ann. **20** (1949), 595–601, MR **11**, 261] the authors prove that, under suitable assumptions, the maximum likelihood method provides a strongly consistent estimate of θ as well as an estimator of G_0 converging to it with probability 1 at all points of continuity. The authors then apply their results to various specific problems, such as those where the location parameter is structural and the scale parameter is incidental or vice versa. The verifications of the assumptions for reasonably wide classes constitutes the main difficulty of the paper. The well-known problem of fitting a straight line to points in the plane when both coordinates are subject to normal error is treated and shown to possess a maximum likelihood solution under the assumptions made in the paper. *A. Dvoretzky.*

Das, M. N. On parametric relations in a balanced incomplete block design. J. Indian Soc. Agric. Statist. **6** (1954), 147–152.

The parameters b, v, r, k, and λ of a balanced incomplete block design satisfy the two basic relationships $bk=rv$ and $r(k-1)=\lambda(v-1)$. The author discusses implications of these relationships for B.I.B. designs and for resolvable B.I.B. designs. *H. J. Ryser.*

Guénot, Robert. Sur une nouvelle définition des interactions dans l'expérimentation à plusieurs facteurs. C. R. Acad. Sci. Paris **244** (1957), 2686–2688.

Pour faciliter l'interprétation des resultats d'expérimentation à plusieurs facteurs, une distinction est établie entre la notion d'interaction moyenne qui intervient dans les tests statistiques habituels et celle d'interaction au sens strict, plus fondamentale sur le plan expérimental. Un théorème permet de relier ces deux notions. *Résumé de l'auteur.*

★ **Statistical Engineering Laboratory of Nat. Bur. Standards. Fractional factorial experiment designs for factors at two levels.** Nat. Bur. Standards Appl. Math. Ser. no. 48 (1957), iv+85 pp.

The experimental plans for fractionally replicated 2^n $(n=5, 6, \cdots, 16)$ factorial experiments are presented for fractional replicates of 2^{-p} for $p=1, 2, \cdots, 8$. The layouts are given in full for fractional replicates of 2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}, and 2^{-5} and are conveniently arranged for use in eliminating either one-way or two-way heterogeneity. The experimental layouts for fractional replicates of 2^{-6}, 2^{-7}, and 2^{-8} are given in full for the first (principal) block with the first treatment given for the remaining blocks.

The plans are useful for many experimental situations where it is necessary to evaluate main effects and some interactions among a number (n) of factors. This compilation makes available a large collection of fractional 2^n factorial designs under one cover. *W. T. Federer.*

Fabian, Václav; and Špaček, Antonín. Experience in statistical decision problems. Czechoslovak Math. J. **6(81)** (1956), 190–194. (Russian summary)

Consider a sequence of decision problems of identical structure, defined by a class of admissible probability distributions of the observations, a class of possible decisions, and a loss function. All relevant spaces are assumed finite. Suppose that the probability distributions of the observations are themselves successive random drawings with an a priori distribution p. The sequence $\{\pi_n\}$ is called a regular sequence estimating p if it converges to p with probability 1 and if, for each n, π_n depends only on the observations in the first $n-1$ decision problems. Then if in the nth decision problem, the Bayes decision corresponding to π_n is used, the actual losses converge $(C, 1)$ with probability 1 to the expected loss corresponding to knowing the true a priori distribution p. *K. J. Arrow* (Stanford, Calif.).

Bennett, B. M. Note on the method of inverse sampling. Trabajos Estadist. **8** (1957), 29–31. (Spanish summary)

The inverse binomial sampling method due to Haldane [Biometrika **33** (1945), 222–225; MR **8**, 477] is extended to the case where the population may change during the sampling procedure. In particular the author obtains the moment generating function for the sample size. *D. M. Sandelius* (Göteborg).

Zindler, Hans-Joachim. Über Faustregeln zur optimalen Schichtung bei Normalverteilung. Allg. Statist. Arch. **40** (1956), 168–173.

Specializing earlier results by Tore Dalenius (Skand.

Aktuarietidskrift, 1950 and 1951), the author considers the problem of sampling a population, say π, of individuals each characterized by the value of a variable X. The purpose of sampling is to estimate the mean μ of X. It is assumed (i) that the distribution of X in π is approximately normal. Also (ii) it is assumed that the population π can be stratified according to the values of X. Specifically, the author considers the stratification into three strata, one with $X \leq \mu - y$, the second with $\mu - y < X \leq \mu + y$ and the third with $\mu + y < X$, where the number y is to be so determined as to insure "optimum statification," i.e., a minimum variance of the mean based on a proportionally stratified sample. The approximate value of y found by the author is 0.61 times the S.D. of the population π. It is shown that (a) with optimum stratification the variance of the estimate of μ amounts to 19 per cent of that corresponding to unrestricted sampling; (b) optimum stratification in four rather than in three strata brings about very little further improvement in accuracy of the estimate; and (c) the combination of optimum stratification with optimum allocation of the sample has little advantage over optimum stratification combined with proportional sampling. The reviewer wonders about practical situations in which the information about π is sufficient for the kind of statification contemplated in (ii) and yet a sampling procedure is required to estimate μ.

J. Neyman (Berkeley, Calif.).

★ **Kolmogorov, A. N. Statistical theory of oscillations with a continuous spectrum.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 12 pp.

Translated from Jubilee Collection, v. 1, Akad. Nauk Press, Moscow-Leningrad, 1947, pp. 242–252; MR 9, 638.

Mackenzie, J. K.; and Thomson, M. J. Some statistics associated with the random disorientation of cubes. Biometrika **44** (1957), 205–210.

Courbis. Sur la détermination des inconnues pour lesquelles on ne dispose que d'observations peu nombreuses et peu précises. Bull. Trimest. Inst. Actuaires Franç. **67** (1956), 187–215.

See also: Burgers, p. 150; Parker, p. 163; Oppenheim and Mazur, p. 191; Mazur and Oppenheim, p. 191; Stewart, p. 204; Aitchison and Silvey, p. 233; Mathematical models of human behavior, p. 233.

PHYSICAL APPLICATIONS

Mechanics of Particles and Systems

Bergmann, Peter G.; Goldberg, Irwin; Janis, Allen; and Newman, Ezra. Canonical transformations and commutators in the Lagrangian formalism. Phys. Rev. (2) **103** (1956), 807–813.

The paper is concerned with the preparation of classical mechanics, in Lagrangian form, for quantization. The authors distinguish between singular or regular Lagrangians for which the determinant $|\partial^2 L/\partial \dot{q}^i \partial \dot{q}^j|$ vanishes or does not vanish. For regular systems they show that Peierls' method of obtaining the commutators [Proc. Roy. Soc. London. Ser. A. **214** (1952), 143–157; MR **14**, 520] is equivalent to that of Bergmann and Schiller [Phys. Rev. (2) **89** (1953), 4–16; MR **14**, 606]. For singular systems Peierls' method fails because there is not a (1, 1) correspondence between infinitesimal canonical transformations and their generators. For any system, the authors define a "large" group of transformations and obtain commutators for its generators. They assert that the subgroup of this which leaves the Lagrangian invariant provides a system of constants of the motion, equivalent to the equations of motion, for which the commutators, obtained above, have properties which suggest they are a suitable starting point for quantization. *A. J. Coleman.*

Novoselov, V. S. Some questions of the mechanics of variable masses, taking into account the internal motion of particles. I. Vestnik Leningrad. Univ. **11** (1956), no. 19, 100–113. (Russian)

In 1897 and 1904, I. V. Meščerskiĭ [Works on mechanics of bodies with variable mass, Gostehizdat Moscow-Leningrad, 1949, pp. 31–182, 214–257] obtained the basic equations of mechanics for variable masses, neglecting the internal motion of the particles. The author of the present paper generalizes these equations to include such internal motion. In addition, the laws of variation of the momentum and the moment of momentum of a system of particles with variable masses are given. These laws are then applied to a rigid body with variable mass. Several illustrative examples are inserted in the text.

E. Leimanis (Vancouver, B.C.).

See also: Leech, p. 165 Yuščenko, p. 197; Hoffmann, p. 226; Popovici, p. 235.

Statistical Thermodynamics and Mechanics

Brittin, Wesley E. Statistical mechanical theory of transport phenomena in a fully ionized gas. Phys. Rev. (2) **106** (1957), 843–847.

Die Gesetze der statistischen Mechanik, die sich auf sich nicht im Gleichgewichte befindenden Systeme beziehen und die besonders von Kirkwood, Irving und Zwanzig ausgearbeitet wurden, werden auf ein System von N geladenen Partikeln angewandt, die ausschliesslich durch ihr elektromagnetisches Feld aufeinander einwirken und die sich in einem grossen Volumen $V = L^3$ befinden. Das auftretende elektrische Feld wird in einem äusseren ($\mathbf{E}^{\text{ext}}$), einem transversalen ($\mathbf{E}^T$) und einem momentan auftretenden Coulombschen Feld ($\mathbf{E}^C$) der vorhandenen Ladungen zerlegt. Ebenso besteht das magnetische Feld aus einem äusseren ($\mathbf{B}^{\text{ext}}$) und einem inneren ($\mathbf{B}^{\text{int}}$) Teil. $\mathbf{E}^T$ und $\mathbf{B}^{\text{int}}$ hängen klassisch durch die Maxwellschen Gleichungen zusammen, der Verfasser beschreibt beide nach Heitler durch eine Reihe von realen Vektorfunktionen $\mathbf{A}_\lambda(\mathbf{r})$. Weiter wird dann so klassisch, wie quantenmechanisch die Hamiltonsche Funktion angegeben, aus der in der ersteren Theorie die Bewegungsgleichungen und in der letzteren die Wellengleichung folgt. In der klassischen Theorie kann man dann eine Verteilungsfunktion f angeben, die der bekannten Liouvilleschen Differentialgleichung $(\partial f/\partial t) + \Omega f = 0$, wo Ω einem Operator bedeutet, genügen muss. Mit Hilfe von f kann dann der mittlere Erwartungswert einer dynamischen Veränderlichen berechnet werden. In der Quantenmechanik kann man ganz analog verfahren. Ähnlich zur klassischen

Verteilungsfunktion wird hier die von Wigner benutzt und mit Hilfe von der wird das quantenmechanische Analogon der Liouvilleschen Gleichung hergeleitet, woraus sich wieder die statistischen Mittelwerte berechnen lassen usw.

Ein wichtiges Resultat der Berechnungen ist, dass für das quantenmechanische gemittelte Feld die Maxwellschen Gleichungen — in ihrer gewöhnlichen Form — gültig sind. Zuletzt werden noch die hydromagnetischen Gleichungen für das betrachtete System berechnet.

T. Neugebauer (Budapest).

Longmire, C. L.; and Rosenbluth, M. N. Diffusion of charged particles across a magnetic field. Phys. Rev. (2) **103** (1956), 507–510.

The diffusion of charged particles across a magnetic field is considered as a problem in Brownian motion in which the guiding centers (of the Larmor orbits of the particles) describe a random-walk process. If the direction of the magnetic field is in the direction of the z-axis, the Boltzmann equations (in the absence of collisions) require that the distribution function be of the form

$$f = f(x+v_y/\omega,\ v_z,\ v_x^2+v_y^2),$$

where $X=x+v_y/\omega$ is the co-ordinate of the guiding center of the particle and $\omega=eB/mc$ is the Larmor frequency. The particular distribution function which the authors assume in their evaluation of the diffusion coefficient is

$$N(X)\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/kT}.$$

If the motion of the guiding centers resulting from the Coulomb encounters between the particles is considered as a Brownian process, the flux $F_1(X_1)$ of guiding centers of particles of type 1 (mass m_1, charge e_1) due to encounters with particles of type 2 (mass m_2, charge e_2) is given by

$$F_1 = N_1(X_1)\langle\Delta X_1\rangle - \tfrac{1}{2}\frac{\partial}{\partial X_1}[N_1(X_1)\langle(\Delta X_1)^2\rangle],$$

where $\Delta X_1 = \Delta v_{1y}/\omega_1$ and

$$(*)\qquad \begin{aligned}\langle\Delta X_1\rangle &= \int d^3v d^3V d\Omega P(v, V, \Omega)\Delta X_1,\\ \langle(\Delta X_1)^2\rangle &= \int d^3v d^3V d\Omega P(v, V, \Omega)\,(\Delta X_1)^2.\end{aligned}$$

In (*) $P(v, V, \Omega)$ is the probability per unit time that a particle 1 with guiding center at X_1 will be involved in a collision with scattering into solid angle $d\Omega$ and in which the relative velocity of encounter is $\mathbf{v}$ and the velocity of the center of mass is $\mathbf{V}$. It is shown that

$$P(v, V, \Omega) = \left(\frac{m}{2\pi kT}\,\frac{M}{2\pi kT}\right)^{3/2} N_2(X_1+\delta)v\sigma(\Omega) \times \exp\{-(mv^2+MV^2)\}/2kT\},$$

where

$$\delta = \frac{C}{B}\left\{\left(\frac{m_2}{e_2}-\frac{m_1}{e_1}\right)V_y - m\left(\frac{1}{e_1}+\frac{1}{e_2}\right)v_y\right\},$$

$\sigma(\Omega)$ is the differential scattering cross section, $M=m_1+m_2$ and $m=m_1m_2/M$. By expanding $N_2(X_1+\delta)$ in a Taylor series in δ and using the Rutherford scattering formula for σ, the authors obtain explicit expressions for $\langle\Delta X_1\rangle$ and $\langle(\Delta X_1)^2\rangle$ and thus solve the basic physical problem in a direct and illuminating manner. *S. Chandrasekhar.*

Oppenheim, I.; and Mazur, P. Density expansions of distribution functions. I. Virial expansion for finite closed systems; canonical ensemble. Physica **23** (1957), 197–215.

Starting from the usual potential energy as a sum of potentials between pairs of particles, the author derives some rather complicated recursion type integro-differential equations involving the partial derivatives of the distribution function with respect to density for a system of N (finite) particles. The λth coefficient in the expansion of the distribution function in powers of the density is shown to be a polynomial of degree λ in $1/N$.

G. Newell (Providence, R.I.).

Mazur, P.; and Oppenheim, I. Density expansions of distribution functions. II. Density expansions in the grand canonical ensemble. Physica **23** (1957), 216–224.

Some of the methods described in part I (reviewed above) are used to investigate expansions of the grand canonical ensemble in powers of the density and activity. The discussion includes both classical and quantum mechanical systems. *G. Newell* (Providence, R.I.).

Glauberman, A. E. A contribution to the general theory of statistical equilibrium of a system of interacting particles. Soviet Physics. JETP 3 (1957), 830–835.

A new method is followed to determine the n-particle distribution functions of a dilute gas in equilibrium as series expansion in powers of the density. It consists in a series expansion solution of the infinite hierarchy of integro-differential equations which the equilibrium distributions satisfy as a consequence of the equations of motion. The determination of this solution makes use of the asymptotic condition that a n-particle distribution function factorizes for large separations in n one-particle distribution functions. The most interesting feature of the author's method is that he does not rely on the canonical distribution usually adopted as starting point for the study of equilibrium properties. *L. Van Hove.*

Rodriguez, A. E. An approximation for the radial distribution function. Proc. Roy. Soc. London. Ser. A. **239** (1957), 373–381.

The author proposes an improved form of the Kirkwood superposition principle in the theory of fluids. The latter consists in approximating the three-particle distribution function n_3 by a product of three two-particle distribution functions n_2. The author adds to this approximate expression two terms involving each a product of two functions n_2, with coefficients determined by two consistency requirements: the dynamical equation relating n_3 to n_2 should be verified in lowest non-trivial order in the density, and the fourth virial coefficient should have its correct value. He leaves out of consideration what seems to the reviewer to be much more obvious consistency requirements: the symmetry of n_3 in its three arguments and the fact that n_2 is directly related to the limit of n_3 when one particle is far from the two other ones. *L. Van Hove* (Utrecht).

Lifshitz, I. M.; and Stepanova, G. I. Vibration spectrum of disordered crystal lattices. Soviet Physics. JETP 3 (1956), 656–662.

This paper gives a systematic treatment of the change in frequency spectrum caused in a crystal by a small concentration of impurities. The case considered is that of

a mixture of isotopes. The method followed is quite general and consists in determining the change in free energy produced by the perturbing isotope as a power series in its concentration. The frequency spectrum is then deduced from the free energy. The general treatment is carried out neglecting the polarization of the lattice vibrations, i.e. treating the atomic displacements as a scalar function. The results including polarization effects are given for the limiting case of a very small mass difference between the isotopes. A special discussion is given of the displacement of the upper end of the spectrum but the authors do not analyze how the singularities in the spectrum are affected. *L. Van Hove* (Utrecht).

Sáenz, A. W. Transport equation in quantum statistics for spinless molecules. Phys. Rev. (2) **105** (1957), 546–558.

A transport equation is derived for a dilute nondegenerate gas of spinless molecules. The derivation makes use of Wigner distribution functions [Phys. Rev. (2) **40** (1932), 749–759] and Kirkwood's time-averaging procedures [J. Chem. Phys. **14** (1946), 180–201]. The physical assumptions are essentially the same as in the derivation of a transport equation by J. Ross and J. G. Kirkwood [ibid. **22** (1954), 1094–1103]. The present transport equation involves a collision integral which is expressed in terms of an exact cross-section for binary collisions. If the latter is replaced by a Born approximation, one obtains the equation of Ross and Kirkwood. *N. Rosen.*

Ginzburg, V. L. On macroscopic theory of superconductivity applicable at all temperatures. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 358–361. (Russian)

The work is based on the equations of V. L. Ginzburg and L. D. Landau [Ž. Eksper. Teoret. Fiz. **20** (1950), 1064–1082]. These equations contain the free-energy density which is not known in the general case, but for which different expressions have been obtained. It is suggested that the free-energy be determined experimentally. A method of doing this is proposed, appropriate calculations being given. *N. Rosen* (Haifa).

See also: Crew, Hill and Lavatelli, p. 183; Gubanov, p. 198; Lifšic and Halatnikov, p. 198; Zubarev, p. 209; Riesenfeld and Watson, p. 215; Bopp, p. 215.

Elasticity, Plasticity

Seth, B. R. New solutions for finite deformation. Proc. Indian Acad. Sci. Sect. A. **45** (1957), 105–112.

The constitutive equation used in this paper is that of a compressible elastic material in which the stress tensor is a linear function of the Green-St. Venant strain tensor. The displacements and the stresses corresponding to the deformation of a thick plate into a spherical shell are determined explicitly. *W. Noll* (Pittsburgh, Pa.).

Thomas, T. Y. Extended compatibility conditions for the study of surfaces of discontinuity in continuum mechanics. J. Math. Mech. **6** (1957), 311–322.

When some dependent variable in a problem of continuum mechanics, e.g. the pressure, is continuous but admits discontinuities of coordinate and time derivatives over a moving surface of discontinuity, the jumps in these derivatives must satisfy well-known conditions of compatibility. In the present paper, more general conditions of compatibility for the first and second derivatives are established without the assumption that the considered variable itself is continuous. A first type of compatibility condition, labelled geometrical, involves only the coordinate derivatives and the geometry of the instantaneous surface of discontinuity; a second type, called kinematical, involves also the time derivatives and the motion of the surface of discontinuity. (The dynamic equations of the considered continuum will, of course, furnish additional restrictions on the discontinuities, but these are not discussed here.) *W. Prager* (Providence, R.I.).

Adkins, J. E.; and Green, A. E. Plane problems in second-order elasticity theory. Proc. Roy. Soc. London. Ser. A. **239** (1957), 557–576.

This paper is a continuation of earlier work by the authors and their coworkers on the approximate solution of problems with plane stress, appropriate to thin plates, or with plane strain in finite elasticity [Adkins, Green, and Shield, Philos. Trans. Roy. Soc. London. Ser. A. **246** (1953), 181–213; Adkins, Green, and Nicholas, ibid. **247** (1954), 279–306; MR **15**, 369; **16**, 765].

The authors investigate various boundary value problems for an infinite body subject to a uniform stress at infinity and with a single inner boundary. They show how all these problems can be reduced to a single coupled boundary value problem for 4 complex potentials involving a total of 4 parameters. They illustrate the method by treating a rigid circular inclusion, a stress-free hole of circular shape before or after the deformation, and a circular hole subject to uniform normal stresses.

W. Noll (Pittsburgh, Pa.).

Paria, Gunadhar. Elastic stress distribution in a three-layered system due to a concentrated force. Bull. Calcutta Math. Soc. **48** (1956), 75–81.

The problem of the stress distribution in a semi-infinite mass having two layers of different material over it due to a concentrated surface force has been solved by the use of Hankel transform of the stress function. A formula for the equivalent depth replacing the two upper layers by the same material as the underlying mass has been derived in the integral form. The ratio of the equivalent depth to the total thickness of the two layers varies as a factor $I^{-\frac{1}{3}}$, which is a function of the thickness-ratio t of the two layers. For $t=0.5$, I has been found to be 1.048.

A table of values or a graphical representation of I for different values of t would have increased the value of the work. *B. R. Seth* (Kharagpur).

Sunčeleev, R. Ya. On a method of solving some boundary problems of the theory of elasticity. L'vov. Politehn. Inst. Nauč. Zap. **30**, Ser. Fiz.-Mat. No. 1 (1955), 3–14. (Russian)

The author applies the generalization of Fourier's method of separation of variables given in his 1955 L'vov dissertation, together with I. M. Gelfand's and Z. Ya. Shapiro's [Uspehi Mat. Nauk (N.S.) **7** (1952), no. 1(47), 3–117; MR **13**, 911; **17**, 875] notion of a differential operator invariant with respect to a given group of transformations, to solve the first and second boundary value problems of the three dimensional theory of elasticity, for a half space, in the case of cylindrical anisotropy [cf. A. Love, Mathematical theory of elasticity, 4th ed., Cambridge, 1927]. *J. B. Diaz* (College Park, Md.).

Aržanyh, I. S. Construction of the integral equations of statics in the theory of elasticity by means of Green's functions. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **10** (1953), no. 2, 5–25. (Russian)

Consider a smooth simple closed surface S in three dimensional space, with interior Q, and let Γ and N denote, respectively, the Green's function and the Neumann function for the solution of the Dirichlet and Neumann boundary value problems for Laplace's equation for $Q+S$. The author remarks that the starting point of this paper was the following question: can the solution of the first and second boundary value problems of the theory of elasticity be computed explicitly in terms of the given data and of the functions Γ and N? To this question the author gives what he terms a negative answer, by deriving only singular and regular integral equations (involving Γ and N) which are satisfied by the displacement vector of the first and second boundary value problems of static elasticity. The first part of the paper is devoted to singular integral equations, whose derivation is based upon a representation (in terms of harmonic functions) of solutions of Lamé's equations of static elasticity. The relationship with earlier results of L. Lichtenstein [Math. Z. **20** (1924), 21–28] is worked out. The second part of the paper is devoted to regular integral equations, whose derivation is based upon an integro-differential equation which is equivalent to Lamé's equation

$$\mu\nabla^2 u+(\lambda+\mu)\text{grad div } u+\rho f=0.$$

Here the relationship to the results of E. and F. Cosserat [C. R. Acad. Sci. Paris **126** (1898), 1089–1091, 1129–1132; **127** (1898), 315–318; **133** (1901), 145–147, 210–213, 271–273, 400, 326–329, 361–364, 382–384] is made explicit.
J. B. Diaz (College Park, Md.).

Aržanyh, I. S. Representation of the dynamical displacement vector by dependent wave functions. Dokl. Akad. Nauk Uzbek. SSR. **1953,** no. 10, 3–5. (Russian. Uzbek summary)

In a previous paper [same Dokl. **1953**, no. 3, 3–7] the author has given representations for the displacement vector u satisfying Lamé's equations of three dimensional elasticity:

$$\alpha \text{ grad div } u-\beta \text{ rot rot } u-\frac{\partial^2 u}{\partial t^2}=F,$$

in terms of independent wave functions. In the present note several other representations are given, involving wave functions which are related by subsidary equations.
J. B. Diaz (College Park, Md.).

Teleman, Silviu. The method of orthogonal projection in the theory of elasticity. Rev. Math. Pures Appl. **1** (1956), no. 1, 49–66.

A translation from the Romanian of the article reviewed in MR **17**, 684.

Csonka, P. Généralisation de la théorie de la torsion de de Saint-Venant. Acta Tech. Acad. Sci. Hungar. **17** (1957), 171–173. (German, English, and Russian summaries)

Referring to one of his previous papers [same Acta **17** (1957), 355–359; MR **19**, 79] the author shows, that Saint-Venant's theory on pure torsion in linear elasticity is also applicable to rods made of a material which does not obey Hooke's law. [On this question one may likewise consult, for instance, V. V. Novozhilov, Foundations of the nonlinear theory of elasticity, Gostehizdat, Moscow-Leningrad, 1948, p. 204; MR **12**, 651; **14**, 924.]
W. Schumann (Zürich).

Položiĭ, G. M. Method of solution of the problem of the bending of prismatic rods. Akad. Nauk Ukrain. RSR. Prikl. Meh. **2** (1956), 257–269. (Ukrainian. Russian summary)

A simple method is given for estimating the maximum stress in a symmetrically bent rod. This method is based on the utilization of the properties of subharmonic functions. The application of the method is illustrated by examples which give estimates for the maximum stress in rods with rectangular cross section with semicircular cutouts. *H. P. Thielman* (Ames, Ia.).

Simons, Roger M. A power series solution of the nonlinear equations for axi-symmetrical bending of shallow spherical shells. J. Math. Phys. **35** (1956), 164–176.

The present paper deals with the calculations of stresses and displacements of thin shallow segments of spherical shells subjected to uniform normal pressure. Two important types of edge restraints (simply supported and clamped edges) are taken into account. There are two essential parameters for this problem: a geometrical parameter μ, which is zero for a flate plate and large ($\mu\gg 1$) for a sufficiently thin shell, and a load parameter γ which is a factor of the nonlinear terms in the differential equations. The linearized differential equations for small deflections can be solved explicitly in terms of Bessel functions [Federhofer, Österreich. Wochenschr. Öffentlichen Baudienst **1916**, nos. 25, 26]. A solution of the problem for small finite deflection, in which case the differential equations of the problem are nonlinear can be obtained in the form of a power series of a coordinate [E. Reissner, Proc. Symposia Appl. Math., v. 3, McGraw-Hill, New York, 1950, pp. 27–52; MR **12**, 557]. A solution has been established by Way [Trans. A.S.M.E. **56** (1934), 627–636], but a serious shortcoming of his series solution is that the boundary conditions satisfied by the solution are known only after the solution is obtained. The only way a prescribed set of boundary conditions can be satisfied is to obtain several power series solutions by choosing two leading coefficients in the power series differently each time and then using interpolation to find the appropriate choices of the leading coefficients to satisfy the prescribed boundary conditions. The interpolation is difficult because there are two independent constants to determine. The principal contribution of this paper is to give a systematic procedure for obtaining the required leading coefficients for the power series solution in order to satisfy the boundary conditions. The technique is given for a shallow shell and contains as a special case the corresponding results for a flat plate. *R. Gran Olsson.*

Vorovič, I. I. On the Bubnov-Galerkin method in the nonlinear theory of nearly smooth shells. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 723–726. (Russian)

Résolution de certains systèmes

$$D_t^2\omega+KD_t\omega+A\omega=F,$$

où ω est un vecteur dépendant de $x\in\Omega$=ouvert de R^n et du temps t, K et A étant des opérateurs différentiels non linéaires (assujettis à des conditions trop longues pour être rapportées ici), et ω étant assujetti à des conditions

initiales et aux limites. On cherche des solutions généralisées. En utilisant des bases on se ramène à des équations différentielles, ici non linéaires (méthode de Faedo-Galerkine-Bubnov). Example. *J. L. Lions* (Nancy).

★ **Vorovich, I. I. On the Bubnov-Galerkin method in the nonlinear theory of shallow shells.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1957. 7 pp.

Translated from Doklady Akad. Nauk SSSR (N.S.) **110** (1956), 723–726; see the paper reviewed above.

Gol'denveĭzer, A. L. More precise theory of the simple boundary effect. Prikl. Mat. Meh. **20** (1956), 335–348. (Russian)

It is well-known that concentrated loads applied at the boundary of a shell produce bending stresses only in the neighbourhood of the boundary or at the point of application. This boundary effect makes it possible to use a simpler approximate procedure in solving the elasticity problem in the boundary zone of a shell, on the condition that the mean surface does not bend. The simplified solution based on this boundary effect for shells of revolution, as given by J. Geckeler [Forschungsarb. Gebiete Ingenieurwesens no. **276** (1926)], was generalized by J. N. Rabotnov [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **47** (1945), 329–331; MR **7**, 142]. Geckeler's and Rabotnov's theories are "theories of first approximation" because in every expansion all terms exept the first are neglected. The author of this paper presents a theory which retains the higher terms in the expansions. The author makes a very detailed error analysis and illustrates his theory by examples of cylindrical, conical and spherical shells.

T. Leser (Aberdeen, Md.).

Ashwell, D. G. The equilibrium equations of the inextensional theory for thin flat plates. Quart. J. Mech. Appl. Math. **10** (1957), 169–182.

The author refers to two papers by Mansfield [same J. **8** (1955), 338–352; MR **18**, 433] and Mansfield and Kleeman [Aircraft Engrg. **27** (1955), 102–108; MR **16**, 974] and gives a further contribution to their recent work on the inextensional theory of thin flat plates. In this simplified large deflection theory the plate is assumed not to be stretched, and therefore deforms into a developable surface, which is completely determined by its edge of regression. In the present paper the differential equation for the edge of regression of the deformed plate is derived from equilibrium and compatibility considerations. In the selected examples of cantilever plates, where the straight built-in edge becomes one of the generators of the developable surface, and also in the examples given by Mansfield, the obtained deformation mode is exact enough, because these types of plates resist against the loads almost entirely by their flexural rigidity. Of course the condition of no stretching of the middle surface is a restriction for the deformation of the plate, and the reviewer would like to know, how in general the boundary conditions have to be — in the sense of an outline for the validity of the inextensional theory — that it approaches the exact large deflection theory with sufficient accuracy.

W. Schumann (Zurich).

Olszak, Wacław; and Mróz, Zenon. Elastic bending of circular plates with eccentric holes. Application of the method of inversion. Arch. Mech. Stos. **9** (1957), 125–153. (Polish and Russian summaries)

Two groups of problems are discussed. In the first, a circular plate containing an eccentric circular hole with both boundaries clamped, is subject to a concentrated load or to certain continuous loadings. Results for a half-plane containing a circular hole are included as a special case. Solution for plane containing two clamped circular holes is also indicated. In the second group similar problems are discussed when one edge of the plate is clamped and the other is simply supported. Numerical results are given for the case of uniform loading. *A. E. Green.*

Colonnetti, Gustavo. L'équilibre des voiles minces hyperstatiques (Le cas des voiles de surface minimum). C. R. Acad. Sci. Paris **243** (1956), 1087–1089.

Colonnetti, Gustavo. L'équilibre des voiles minces hyperstatiques. (Le cas des voiles de surface minimum). C. R. Acad. Sci. Paris **243** (1956), 1701–1704.

As the author notes, the general solution of the equations of equilibrium for a shell referred to geodesic co-ordinates was obtained by Storchi [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7** (1949), 227–231; **8** (1950), 116–120, 326–331; MR **11**, 556, 757; **12**, 219]. The author considers minimal surfaces referred to their lines of curvature. In the first note he assumes that the order of covariant differentiations is invertible; in the second, he notes that this is generally false. By straightforward elimination he reduces the problem to solution of a single partial differential equation for the shear resultant, but he does not solve this equation. {Reviewer's note: The general solution in general co-ordinates for a surface of revolution was obtained by L. Finzi [ibid. **20** (1956), 205–211, 338–342] his proof is faulty but his result is true. L. Finzi indicates the equivalence of the problem of solving $s^{km}{}_{,m}=0$, $s^{km}=s^{mk}$, in any space with the problem of finding conditions of compatibility for the system $v_{(k,m)}=d_{km}$ in the same space.} The author applies these results to his own variational theory of elasto-plastic media. He concludes that the normal resultants are constant if and only if they vanish.

C. Truesdell (Bloomington, Ind.).

Nazarov, A. A. Equations of equilibrium of gently slanting shells and their application. Akad. Nauk Ukrain. RSR. Prikl. Meh. **2** (1956), 270–283. (Ukrainian. Russian summary)

This paper presents the solution of problems on the equilibrium of gently sloping shells supported along the perimeter of a rectangle. (By "gently sloping" it is understood that the normal to the shell surface does not make an angle greater than 15° with the normal to the plane of the supporting rectangle.) Results of numerical computations for the deflection and stress are given. The concept of a sloping shell with a given plan (support contour) is defined. Such shells have constant curvature parameters, while their coordinate curves $x_1=$const, $x_2=$const are lines of curvature of the smoothed (approximating) surface. Equations of equilibrium are derived for sloping shells with a given plan. These equations are solved in terms of trigonometric series for a shell rectangular in plan. Certain limitations on the applicability of the developed theory of sloping shells are pointed out at the end of the paper. *H. P. Thielman* (Ames, Ia.).

★ **Donnell, L. H. A theory for thick plates.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 369–373. American Society of Mechanical Engineers, New York, 1955. $9.00.

The paper outlines a solution procedure for the differ-

ential equations of linear elasticity for a plate, that is for a region bounded by two parallel planes $z=\pm\frac{1}{2}h$. The solution consists of terms containing successively higher powers of z and successively higher derivatives of the load functions $p(x, y)$ associated with the surfaces $z=\pm\frac{1}{2}h$. The procedure would seem to be appropriate in the event that the plate is of infinite extent and the surface load functions have derivatives of any order, everywhere. [For this class of problems it is also possible to obtain an explicit solution by means of the Fourier integral, however, without making any differentiability assumptions.] The problem of the plate without surface loads but acted upon by edge loads is mentioned but not treated.

E. Reissner (Cambridge, Mass.).

Koppe, Eberhard. Die dicke Platte mit nichtlinearer Spannungsverteilung. Z. Angew. Math. Mech. **37** (1957), 38–44. (English, French and Russian summaries)

The author presents a system of series solutions for the differential equations of three-dimensional linear elasticity, for the purpose of solving boundary value problems for a body bounded by two planes $z=\frac{1}{2}h$ and by a cylindrical surface $f(x, y)=0$. It is implied that the system of solutions which is obtained is general enough to allow an exact treatment of all significant thick-plate problems, including the problem of the edge conditions along edges $f(x, y)=0$. The brevity of the presentation prevents the reviewer from arriving at a definite opinion in regard to this property of the author's series. If the solution is in fact as general as indicated then the problem would seem to be simpler than heretofore believed by many students of the subject.

E. Reissner (Cambridge Mass.).

Bycroft, G. N. Forced vibrations of a rigid circular plate on a semi-infinite elastic space and on an elastic stratum. Philos. Trans. Roy. Soc. London. Ser. A. **248** (1956), 327–368.

The types of motion which are considered are, (1) vertical translation, (2) rotation about a horizontal axis, (3) rotation about a vertical axis, (4) horizontal translation. In no case, except for the static problems (1), (2) and (3) for the half space, are exact closed-form solutions of the mixed boundary value problem possible. For problem (3) for the half space there exists an exact series solution for the dynamic problem. The authors method in problems (1) to (3) consists in the use of the exact static pressure distribution under the rigid plate for the purpose of deducing ingenious upper and lower bounds for the impedance of the rigid plate. In problem (4) additional simplifying assumptions are made in order to arrive at an approximate solution. Numerical determination of the functions of interest involves the evaluation of complicated Fourier-Bessel integrals. It is stated that agreement between theoretical work and experimental work (with the help of a foam rubber block 3 feet square and 14 inches deep, this work to be reported separately) was close.

E. Reissner (Cambridge, Mass.).

★**Писаренко, Г. С. [Pisarenko, G. S.] Колебания упругих систем с учетом рассеяния энергии в материале. [Vibrations of elastic systems taking account of dissipation of energy in the material.]** Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1955. 238 pp. 12.50 rubles.

The book consists of two parts. Part I gives the theory, Part II describes experimental procedures. In Part I the author develops the theory of forced vibrations of elastic systems when the dissipation of energy in the medium is taken into account. He assumes that the dissipation is caused by local plastic deformations in the material and that these deformations increase with the increasing load, which means that they increase with the increasing amplitude. This assumption is based on experimental evidence. Consequently the differential equations controlling such vibrations are non-linear. Materials which are considered in this book have a narrow hysteresis loop, that is both branches of the hysteresis loop differ little from straight lines of Hooke's law; hence the author is justified in introducing a small parameter in his non-linear differential equations. He solves approximately these equations using the Krylov-Bogoliubov method of asymptotic expansions in powers of the small parameter and introduces his own method for computing the resonance curves. By procedures described above the author solves several important problems of vibrations of turbine blades. A number of typical problems are solved numerically, which involve experimental constants characterizing deformation properties of the materials. In Part II the author describes in detail techniques of determining these constants. The book embraces a number of author's original researches which are listed in the bibliography. The book is written simply and clearly and it is very easy to understand with a background of advanced calculus.

T. Leser (Aberdeen, Md.).

Vacca, Maria Teresa. Vibrazioni torsionali di un cilindro circolare di lunghezza finita. Atti Sem. Mat. Fis. Univ. Modena **7** (1953–54), 87–104 (1956).

The problem, indicated in the title leads (through various simplifying assumptions including cylindrical symmetry) to a single linear partial differential equation of the second order. It is solved under the following three conditions: 1) Assigned initial displacements and velocities on the bases, and free lateral surface; 2) zero initial displacements and velocities on the bases, and assigned tangential forces on the lateral surface; 3) assigned initial displacements and velocities on the bases, and assigned tangential forces on the lateral surface. In cases 1) and 3) the displacements (and velocities) on the two bases are assumed to be equal in magnitude but of opposite sign. Case 1) is handled by separation of variables, case 2) by Laplace transform theory, and case 3) is a combination of the other two. The solutions (which are clearly not uniquely defined by these conditions, since one should also be able to assign initial displacements and velocities on the lateral surfaces at least, if not actually at all interior points as well) involve infinite series of Bessel and trigonometric functions. Although the results may be new, the methods are completely routine.

D. C. Lewis, Jr. (Baltimore, Md.).

Grigolyuk, È. I. On small oscillations of thin elastic shells. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1956**, no. 6, 35–44. (Russian)

Non-axisymmetric vibrations of conical shells have been studied since the first appearance of loudspeakers. M. J. O. Strutt solved the problem neglecting changes in curvature in meridional direction and assuming inextensibility of the middle surface [Ann. Physik (5) **17** (1933), 729–735]. A. T. van Urk and G.B. Hut measured experimentally vibrations of a conical shell [ibid. **17** (1933) 915–920]. The measured frequencies differed from Strutt's computed frequencies by a factor of 2 to 3. The assump-

tion of inextensibility made by Strutt turned out to be not realistic.

This paper is a development of the author's previous investigations on conical shells [Inžen. Sb. **19** (1954), 73–82; Calculation of spatial constructions, v. 3, Gosstročizdat, Moscow, 1955, pp. 375–420], where the extensibility of the middle surface was taken into account. The author solves the problem by the energy methods but the axisymmetric case cannot be obtained from his solution. A cylindrical shell becomes a special case, which agrees with results obtained by several other investigators. The author's results disagree also with Urk's experiments, but he claims that this is due only to different boundary conditions. *T. Leser* (Aberdeen, Md.).

Chakravorty, J. G. Some problems of plane strain in a cylindrically aeolotropic cylinder. Bull. Calcutta Math. Soc. **47** (1955), 231–234.

Carrier [J. Appl. Mech. **10** (1943), A-117-A-122] has developed the approximate theory of stretching of a cylindrically aeolotropic plate by forces acting in the plane of the plate. This theory has been applied by A. M. Sengupta to some problems of vibration of plates of the same material [Indian J. Theoret. Phys. **1** (1953), 125–132; MR **16**, 1070]. As the exact determination of stresses satisfying the equations of equilibrium and the conditions of compatibility is very difficult, both authors have considered only average stresses and displacements in the plate. The present author has obtained exact solutions of some three-dimensional problems of a cylindrically aeolotropic material [Bull. Calcutta Math. Soc. **48** (1956), 171–176]. In the present paper an exact solution of the problem of torsional vibration of a cylinder of cylindrically aeolotropic material has been obtained when the cylinder is either hollow or has an isotropic core.

R. Gran Olsson (Trondheim).

Aržanyh, I. S. Stress tensor functions for the dynamics of an elastic body. Dokl. Akad. Nauk Ŭzbek. SSR. **1953**, no. 7, 3–4. (Russian. Uzbek summary)

Using the method empoyed in a previous paper [Dokl. Akad. Nauk SSSR (N.S.) **83** (1952), 195–198; MR **13**, 1000] the author obtains a representation for solutions of the dynamical equations of an elastic body (three displacement components and six stress components) in terms of twenty one arbitrary, sufficiently differentiable functions of x, y, z, t. The present formulas reduce to the known Maxwell-Morera representation formulas in the static case [cf. A. Krutkov, Stress tensor functions and general solutions in the static theory of elasticity, Izdat. Akad. Nauk SSSR, Moscow, 1949]. *J. B. Diaz.*

Barta, J. Über die Stabilität des Gleichgewichtes eines gedrückten Stabes von veränderlichem Querschnitt. Acta Tech. Acad. Sci. Hungar. **17** (1957), 305–310. (English, French and Russian summaries)

The author furnishes bounds for the buckling load P_k of a thin, simply-supported, elastic column of variable cross-section. The principal result of the paper states that if $u(x)$ is a function having suitable differentiability properties, and if $u\to-\infty$ as $x\to0$, $u\to\infty$ as $x\to l$, then

$$[(u'-u^2)EJ]_{\min}\leq P_k\leq[(u'-u^2)EJ]_{\max}.$$

Columns of constant and linearly varying cross-section are considered as examples. *W. E. Boyce* (Troy, N.Y.).

Barta, J. Eine Modifikation des Vianelloschen Iterationsverfahrens. Acta Tech. Acad. Sci. Hungar. **17** (1957), 341–347. (English, French and Russian summaries)

The paper describes an iteration method for determining the buckling load of a simply-supported column of variable cross-section subjected to axial compression. The general step consists of solving the initial value problem

$$EJY_{kn}''+P_{kn}Y_{kn}=0,$$
$$Y_{kn}(0)=0,\quad Y_{kn}'(0)=1,$$

rather than a related boundary value problem, as in Vianello's method. The next value of P_{kn} is found from the formula

$$P_{k(n+1)}=P_{kn}+6EJ_{\min}Y_{kn}(l)l^3.$$

It is shown that $\lim_{n\to\infty}P_{kn}=P_k$, the critical load; moreover, $P_{kn}\leq P_k$ for all n. *W. E. Boyce.*

Csonka, P. Modifikation des Barta'schen Iterations-Verfahrens zur Bestimmung der Knickkraft gerader Druckstäbe. Acta Tech. Acad. Sci. Hungar. **17** (1957), 349–353. (English, French and Russian summaries)

The author refines Barta's iteration method [see the paper reviewed above] for the computation of the buckling load of a simply supported column of varying cross-section by introducing a parameter which can be chosen to speed the convergence of the process. Specifically, it is shown that the $(n+1)$st approximation to the critical load can be computed from the nth by the formula

$$P_{k,n+1}=P_{k,n}+6\alpha EJ_{\min}Y_{kn}(l)/l^3,$$

where α is a function of l, $EJ_{\max}$, and $P_{k,n}$. The most significant improvement occurs for columns of nearly constant cross-section; in the extreme case $\alpha=\pi^{2/3}$ compared to $\alpha=1$ in Barta's method. *W. E. Boyce.*

Skuridin, G. A. On the theory of scattering of elastic waves on curvilinear boundary. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1957**, 161–183. (Russian)

The present work contains an excellent analysis of the reflection of plane elastic waves on curvilinear boundaries. In the introduction the author presents a review of various theories pertaining to the reflection and diffraction of acoustic, electro-magnetic and elastic waves by different configurations. Thereafter, the author puts forth his solution to the problem, which is based on the Kirchhoff-type approximations. The approximations follow from the condition $kp\gg1$ (k=wave number in question). Both longitudinal and transversal waves are considered. The solutions are obtained in terms of infinite integrals. These integrals are evaluated (approximately) by means of the method of the stationary phase. Particular attention is paid to the principal wave. Some calculated results are also presented. The author makes reference to fifty-two contributions. It seems that more accurate asymptotic expansions of the deduced formulas might cover a greater frequency-range pertaining to the diffraction or reflection of elastic waves on not too strongly curved boundaries. *K. Bhagwandin* (Oslo).

Filippov, A. F. Certain problems of difffraction of plane elastic waves. Prikl. Mat. Meh. **20** (1956), 688–703. (Russian)

The author obtains approximate solutions to a class of problems related to the diffraction of elastic waves in homogeneous and isotropic media. He does not, however,

present any general theory; as a matter of fact, he confines his analysis to the diffraction of the waves by a cut. The analysis is based on boundary-value problems of complex function theory. (This type of analysis is, however, not entirely new.) In order to be able to obtain the solutions in question, the author introduces new variables which are more convenient to deal with. As an illustration of the developed theory, he presents numerical values of the velocity components, the angle of attack being 45 degrees, and the time taken to be unity. The latter part of the paper deals with asymptotic expansions. The author's analysis is, however, not rigorous; proofs are not presented, neither are the regions of validity in the complex plane properly established. Nevertheless, the analysis does seem physically reasonable, although a good deal of additional investigation is warranted.

K. Bhagwandin (Oslo).

Juščenko, A. A. On longitudinal vibrations of a thread with a variable mass on one end. Ukrain. Mat. Ž. **8** (1956), 460–462. (Russian)

In two previous papers the author [Dopovidi Akad. Nauk Ukrain. RSR **1955**, 529–532; **1956**, 235–237; MR **17**, 917; **18**, 527] has obtained the partial differential equations of vibration of an elastic and elastic-viscous hoisting thread under the assumption that the mass attached at one end of the thread varies according to a linear law with respect to the time t. Using a method of Yu. D. Sokolov [ibid. **1955**, 21–25; MR **17**, 307] the solution for $u(x, t)$, the absolute lengthening of a segment of the thread of length x is sought in the form

$$u(x, t)=x\psi(t)+x^2\varphi(t).$$

In this way the problem is reduced to the solution of two ordinary second order differential equations with variable coefficients and with zero initial conditions.

The present paper investigates numerically the influence of viscous friction η and of the velocity of variation of the mass (time of loading t^*) on the dynamic pull caused in the thread during the lifting of a load. The above mentioned differential equations are solved for various values of the parameters η and t^* by the use of the small electronic computer of the Academy of Sciences of the Ukrain. SSR. For the results obtained, which are of a technical nature, the interested reader is referred to the original paper.

E. Leimanis (Vancouver, B.C.).

Babič, V. M. Hadamard's method in the dynamics of a nonhomogeneous elastic medium. Vestnik Leningrad. Univ. **11** (1956), no. 1, 107–124. (Russian)

The author considers the Cauchy problem (prescription of the displacements and their partial derivatives with respect to time, at $t=0$, for all x_1, x_2) for the system of dynamical equations of a non-homogeneous plane elastic medium [see S. G. Mihlin, Prikl. Mat. Meh. **11** (1947), 423–432; MR **9**, 255]. The method followed consists in constructing a certain matrix, called the "singular part of Green's matrix", which enables the author to prove that if a twice continuously differentiable solution to the Cauchy problem exists, then this solution satisfies an integral equation of Volterra type, whose solution can be shown to be unique by the method of successive approximations. [For this procedure in the case of a single hyperbolic partial differential equation, see J. Hadamard, Le problème de Cauchy ···, Hermann, Paris, 1932, p. 132 ff., p. 415 ff.; and S. L. Sobolev, Trudy Seĭsmol. Inst. no. **6** (1930).] The proof of the existence of the solution, starting with the Volterra integral equation mentioned, is said to follow the same lines as the corresponding proof in the paper of Sobolev just alluded to.

J. B. Diaz.

Ying, C. F.; and Truell, Rohn. Scattering of a plane longitudinal wave by a spherical obstacle in an isotropically elastic solid. J. Appl. Phys. **27** (1956), 1086–1097.

Scattering cross-sections have been obtained for the scattering by a spherical obstacle of a plane longitudinal wave propagating in an unbounded isotropically elastic solid. Three different types of obstacles are considered: an isotropically elastic sphere, a rigid sphere, and a spherical cavity. For Rayleigh scattering (small enough spheres compared to incoming wave length) it is shown, for the cases of the isotropically elastic sphere and the spherical cavity, that the scattering cross-section can be expressed in a form similar to the corresponding scatter of a plane longitudinal wave by an inviscid fluid sphere in an inviscid fluid; namely, Gk^4a^6. G is a non-dimensional scatter coefficient which depends upon the ratio of the longitudinal wave velocity to the transverse wave velocity in the materials involved, k is the wave number of the incoming wave, and a is the radius of the spherical obstacle. The form of the rigid sphere onstacle is Ga^2, i.e., independent of the wave length of the incoming wave. Comparisons are made among the various scatter coefficients.

P. Chiarulli (Chicago, Ill.).

Craggs, J. W. The propagation of infinitesimal plane waves in elastic-plastic materials. J. Mech. Phys. Solids **5** (1957), 115–124.

The equation for wave propagation in a three-dimensional elastic-plastic medium are written down for an isotropic work hardening law relating the Mises-Henky yield limit and the plastic work. The propagation of a plane wave involving infinitesimal discontinuities is investigated when the undisturbed material is at the yield point. Two wave velocities are determined which depend on the relation between the initial stress tensor and the increments. These speeds vary from elastic values to lower plastic values, and only in special circumstances is the one-dimensional "tangential modulus" speed predicted. Interactions with unloading waves are discussed.

E. H. Lee (Providence, R.I.).

Yakovleva, G. F. Conditions for the periodicity of forced longitudinal, transverse and critical vibrations of a rod taking account of the after effect. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **16** (1955), 126–138. (Russian)

The "after effect" stress-strain relations employed are those of Boltzmann-Volterra [see V. Volterra, Theory of functionals and of integral and integro-differential equations, Fac. Ci., Univ. Central, Madrid, 1927] and the integro-partial differential equations used for the displacement in the three particular problems considered are those given by M. I. Rozovskiĭ [Izv. Akad. Nauk SSSR, Otd. Tehn. Nauk **1948**, 601–622; MR **10**, 88]. The author's conditions for the existence of periodic vibrations are based on four theorems (to be proved in full later) concerning the solutions $z(x)$ of the non-homogeneous integro-differential equation

$$L_n(z)+\int_a^x \sum_{j=1}^{l} P_j(x-t+a)e^{-\beta_j(x-t+a)}M_m(z)dt=f(x),$$

where $L_n(z)$ and $M_m(z)$ are given linear homogeneous

differential operators with constants coefficient, of orders n and m, respectively; the P_j are given polynomials of degree μ_j; the β_j are complex constants; and $f(x)$ is a given periodic function of period 2ω, which changes sign every half period. Theorems concerning the existence and the explicit representation of the solution of the corresponding homogeneous integro-differential equation had been given earlier by Ya. V. Bykov [Kirgiz. Gos. Univ. Trudy Fiz.-Mat. Fak. **1953**, no. 2, 67–83; MR **17**, 750].

J. B. Diaz (College Park, Md.).

Finzi, Leo. Formulazioni variazionali della congruenza nei corpi elastoplastici. Rend Sem. Mat. Fis. Milano **26** (1954–55), 25–44 (1957).

This text of a lecture delivered in 1955 presents a survey of certain variational principles of mechanics of solids. The principles correspond to the principle of minimum complementary energy in linear elasticity. Nonlinear elastic; rigid, perfectly plastic; and elastic, perfectly plastic behavior are considered. *W. Prager.*

Kačanov, L. M. Stability of thin-walled bars under elastic-plastic deformations. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 803–806. (Russian)

For a thin-walled bar of open cross-section, loaded axially, an equation given by Hill [The mathematical theory of plasticity, Oxford, 1950; MR **12**, 303] is reduced to $\delta\sigma_z = E'\delta\varepsilon_z$, where E' is the tangent modulus for the region of strain under consideration. Author develops his criterion for stability from the average tangent modulus, $\bar{E} = h^{-1}\int_0^h E'dn$, taken over the wall thickness. A special case gives a result coinciding with a result of A. R. Flint [J. Mech. Phys. Solids **1** (1953), 90–102].

R. E. Gaskell (Seattle, Wash.).

Bychawski, Zbigniew. Resolving kernel of the Volterra equation in the case of the generalized creep function. Arch. Mech. Stos. **9** (1957), 247–257. (Polish and Russian summaries)

This paper contains remarks about the solution of a Volterra integral equation coming up in the theory of creep in prestressed reinforced concrete. *W. Noll.*

Olszak, Wacław; and Urbanowski, Wojciech. The plastic potential and the generalized distortion energy in the theory of non-homogeneous anisotropic-elastic-plastic bodies. Arch. Mech. Stos. **8** (1956), 671–694.

The authors note that the Mises yield condition for isotropic bodies may be interpreted as a condition that the energy of distortion be limited to a given value. For anisotropic bodies, the total energy is not as obviously separable into a distortion energy and a volume change energy. Different results are obtained depending upon whether the volume change energy is defined as that produced by a pure hydrostatic tension $\sigma_{ij} = \sigma\delta_{ij}$ or by a uniform elongation $\varepsilon_{ij} = \varepsilon\delta_{ij}$. Yield conditions are established for both hypotheses.

In the most general case, the body will be both elastically and plastically anisotropic. Of particular interest is a special case when the plastic coefficients of anisotropy can be expressed in terms of the elastic coefficients plus one additional scalar quantity (the yield stress). The authors suggest adoption of assumption to this effort. Experiments could then determine which generalization of the distortion energy was more appropriate. This would involve determination of 15 material constants.

Allowance is made throughout for possible material variation in space. This inclusion of non-homogenity does not appear to affect the analysis. *P. G. Hodge, Jr.*

See also: Laval, p. 199; Krasil'nikov, p. 207; Baños, p. 210.

Structure of Matter

Gubanov, A. I. Electron scattering from thermal vibrations in a liquid. Soviet Physics. JETP **4** (1957), 465–473.

Adopting a lattice model to describe the short-range order in liquids, Gubanov assumes that (i) Bloch functions can be used to approximate the wave functions of electrons in liquids, (ii) the phonon wave functions are the same in the liquid and crystalline states, and (iii) the interaction between electrons and phonons can be treated as a perturbation. If momentum were conserved in electron-phonon interactions, the mean free path of electrons should be the same in liquids and crystals, under these assumptions. But, because of the absence of long-range order in a liquid, Gubanov finds that momentum is not conserved, i.e., that there is a reaction on the liquid in general. The consequent scattering of electrons he calls 'phonon-liquid' scattering.

First taking only acoustic frequencies into account, explicit approximations are obtained for the electronic mean free path, when the temperature is either high or low compared with the Debye temperature. At low temperatures metals and semi-conductors have to be considered separately. Results are given of similar calculations taking optical frequencies into account. It is concluded that the phonon-liquid scattering is unimportant in semi-conductors, but may be important in liquid metals. *H. S. Green* (Dublin).

Lifšic, E. M.; and Halatnikov, I. M. Hydrodynamics of liquid helium. Nuovo Cimento (10) **4** (1956), supplemento, 735–742.

Daunt and Smith [Rev. Mod. Phys. **26** (1954), 172–236] recently reviewed on an equal basis 8 different equations, which have been proposed by different authors for the motion of the superfluid in liquid helium below the λ-point. This has moved the authors of the paper now reviewed to re-state the system of equations which they have developed in association with Landau, and to affirm that no other system is consistent with certain accepted principles and properties.

They first neglect dissipative processes, and derive a set of equations, all but two of which are consistent, in this approximation, with those adopted by the majority of other authors. The exceptions arise from the distinctive assumption, due originally to Landau, that the motion of the superfluid is exactly irrotational.

Lifšic and Halatnikov go on to consider dissipative processes, again assuming that the superflow is irrotational. They do not include in their equations a term, considered necessary by some authors, proportional to the rate of creation of superfluid. They conclude with a brief discussion of the application of this theory to mixtures of He^3 and He^4. *H. S. Green* (Dublin).

★**Herman, Frank. Theoretical investigation of the electronic energy band structure of solids.** International conference on current problems in crystal physics. pp. 105–126. Massachusetts Institute of Technology, Cambridge, Mass., July 1–5, 1957.

Die bisherigen Ergebnisse der Bändertheorie der festen Körper werden kurz zusammengefasst und die weiteren Entwicklungsmöglichkeiten dieses Modells werden andeutungsweise besprochen. Dabei wird bloss die Erklärung der physikalischen Eigenschaften des ideal reinen und unverletzten Kristalls mit Hilfe des Bändermodells berücksichtigt, die sogenannten strukturempfindlichen Eigenschaften liegen also ausserhalb der Rahmen der Arbeit.

Nach der Born-Oppenheimerschen Näherung kann man in vielen Fällen die Bewegungen der Atomkerne und die der Elektronen voneinander separieren, das Bändermodell ist eigentlich nur in diesem Falle anwendbar. Weiter wird das Problem der Zurückführung der im Kristall auftretenden Vielelektronenprobleme auf ein Einelektronenproblem — das ja praktisch mathematisch allein behandelbar ist — besprochen und die Unentbehrlichkeit der gruppentheoretischen Methode bei der Bestimmung der Symmetrieeigenschaften der Wellenfunktion im Kristall und der Energiefunktion wird betont. Als Beispiel der auftretenden Verhältnisse wird die Bänderstruktur des Siliciums und des Germaniums nach den eigenen Untersuchungen des Verfassers graphisch dargestellt. *T. Neugebauer* (Budapest).

Laval, Jean. L'élasticité du milieu cristallin. I. L'énergie potentielle d'un cristal et les constantes de rappel atomiques. J. Phys. Radium (8) **18** (1957), 247–259.

Der Verfasser arbeitet ganz allgemein die Theorie der bei der Deformation eines Kristallgitters zwischen den einzelnen Atomen auftretenden Kräfte aus. Auf das quantenmechanische Wesen dieser Kräfte wird dabei, abgesehen von einzelnen ganz allgemeinen Bemerkungen, nicht eingegangen, sondern die zwischen zwei Atome wirkenden Kräfte werden bloss formal unter der Annahme eingeführt, dass auch zwischen den einzelnen Bausteinen der Materie das Hookesche Gesetz gültig sein muss. Die Grundannahmen des Verfassers sind also zu denen von M. Born [Dynamik der Kristallgitter, Teubner, Leipzig-Berlin, 1915] weitgehend analog. Für die auf ein Atom im deformierten Gitter wirkende Kraft folgt dann unter diesen Annahmen

$$(1)\qquad F_{j\,\alpha}^{m}=\sum_{pk}\sum_{\beta} C_{j\,\alpha\,k\,\beta}^{m-p}(u_j^{m\beta}-u_k^{p\beta})\quad(p+k\neq m+j),$$

wo die u die Verrückungen der Atome aus ihren ursprünglichen Lagen und die C Konstanten bedeuten. Summiert wird über alle übrigen Atome des Kristalls. (Die α und β bedeuten die Koordinatenachsen.) (1) wird zuerst auf eine einfachere Form gebracht und danach wird die Theorie der thermischen Dilatation ausgearbeitet, wobei im Ausdruck der potentiellen Energie (jedoch nur in diesem Falle) nicht nur die quadratischen, sondern auch noch die Glieder dritten Grades berücksichtigt werden. Danach bespricht der Verfasser das Problem der potentiellen Energie des Kristallgitters und leitet weiter wichtige Zusammenhänge zwischen den eingeführten Konstanten her. Ganz allgemein wird ausserdem gezeigt, dass im Kristallgitter die Dreikörperkräfte immer nach den zwischen nur zwei Atome auftretenden Wechselwirkungen entwickelbar sind. Analoge Verhältnisse treten auch bei Vierkörperkräften auf usw. Im letzten Teil der Arbeit werden die Hypothesen der zentralen und nichtzentralen Kohäsionskräften besprochen; bezüglich des letzteren Falles beruft sich der Verfasser auf die quantenmechanischen Berechnungen von S. O. Lundquist [Ark. Fys. **9** (1955), 435–456]. Zuletzt werden noch zu den bekannten Resultaten von Born einzelne ergänzende Bemerkungen gemacht. *T. Neugebauer* (Budapest).

Atoji, Masao. The integral transformations of atomic scattering factors and their applications. Acta Cryst. **10** (1957), 291–303.

Bekannterweise kann man in der Theorie der Streuung von Röntgenstrahlen die im Atom gebundenen Elektronen zwar in erster Näherung als frei betrachten, doch ist der gestreute elektrische Vektor demzufolge noch nicht einfach zur Zahl der Elektronen proportional, weil zwischen den gestreuten Vektoren Phasenunterschiede auftreten, die man durch Einführung des Atomformfaktors

$$(1)\qquad f(s)=\int_0^\infty 4\pi r^2\rho(r)\frac{\sin 2\pi sr}{2\pi sr}\,dr,$$

wo $s=2\sin\theta/\lambda$ ist, berücksichtigt. $\rho(r)$ bedeutet hier die Elektronendichte, λ die Wellenlänge und θ den Streuwinkel. Durch Umkehrung von (1) kann man $\rho(r)$ berechnen, wenn die Funktion $f(s)$ empirisch bekannt ist, oder richtiger ausgedrückt, kann man durch eine Fouriersche, Hankelsche und Gausssche Transformation die radiale Elektronendichte und deren zwei- und eindimensionale Projektionen berechnen. Dieses Ergebnis ist noch ergänzungsbedürftig, weil die Formeln noch mit dem Debye-Wallerschen Temperaturfaktor $\exp(-Bs^2/4)$, wo $B=8\pi^2\bar{u}^2$ ist und $\bar{u}^2$ den quadratischen Mittelwert der thermischen Verrückung bedeutet, ergänzt werden müssen. Der Verfasser leitet erstens eine halbempirische Formel für den Atomformfaktor her, nach der

$$(2)\qquad f(s)=\sum_n \frac{H}{(s^2+E)^n}$$

ist und wo die H und E von s unabhängig sind (jedoch für verschiedene n nicht die selben Werte haben). Weiter werden die erwähnten Elektronendichten, deren Werte im Mittelpunkt und die Krümmungsradien der erhaltenen Kurven ebenfalls im Mittelpunkt, erstens mit Hilfe von Wasserstoffeigenfunktionen, dann für höhere Atome mit die halbempirischen analytischen Eigenfunktionen von Slater und zuletzt nach der (statistischen) Thomas-Fermischen Methode berechnet. (Letztere ist auch bei leichten Atomen anwendbar, wenn nur die thermische Unruhe genügend gross ist.) Die erhaltenen Resultate werden auch graphisch dargestellt, dabei sind die Abhängigkeiten der Anfangsdichten von dem Werte von B (thermische Bewegung) besonders interessant. Die numerischen Resultate werden auch mit der Erfahrung verglichen und auf weitere Anwendungsmöglichkeiten wird aufmerksam gemacht. *T. Neugebauer* (Budapest).

Ino, Tadashi. Studies on the radial distribution analysis in diffraction methods. I. Errors due to the approximation of integration by summation. J. Phys. Soc. Japan **12** (1957), 495–499.

In the calculation of a radial distribution function $rD(r)=(2\pi^2)^{-1}\int_0^\infty sI(s)\sin sr\,ds$ the integral usually is approximated by a sum as

$$rD_\Sigma(r)=\frac{\Delta s}{2\pi^2}\sum_{n=1}^{\infty} n\Delta s\cdot I(n\Delta s)\cdot\sin(n\Delta sr).$$

In the paper the mathematical property of $rD_\Sigma(r)$ is derived and the error due to the approximation is proved to be expressed as

$$rD_\Sigma(r)-rD(r)=\sum_{m=-1,\neq 0}^{\infty}{}'\left(\frac{r+2m\pi}{\Delta s}\right)D\left(\frac{r+2m\pi}{\Delta s}\right).$$

Furthermore, the formula for the estimation of the error is derived. It turned out that in order to obtain a high accuracy from the approximation one must make the spacing Δs less than π/R_0, where R_0 is the distance beyond which $rD(r)$ is regarded practically zero. *W. Nowacki.*

Fick, E. Die Termaufspaltung in elektrostatischen Kristallfeldern. Z. Physik **147** (1957), 307–316.

In einem rein elektrostatischen Kristallfeld ergibt sich für die Betheschen Kristallterme teilweise eine zusätzliche Entartung, die durch die Invarianz des Hamilton-Operators gegenüber einer Umkehr aller Impuls- und Spinrichtungen bedingt ist. Durch eine Untersuchung der irreduziblen Darstellungen der 32 Kristallklassen werden die zusammenfallenden Energieterme bestimmt. Die Anzahl und Vielfachheit der Terme, die sich innerhalb eines Kristallsystems als unabhängig von der Kristallklasse ergeben, werden angegeben. In einem Anhang sind die spezifischen, irreduziblen Darstellungen der Kristalldoppelgruppen zusammengestellt. *W. Nowacki.*

Fluid Mechanics, Acoustics

Murray, J. D.; and Mitchell, A. R. Flow with variable shear past circular cylinders. Quart. J. Mech. Appl. Math. **10** (1957), 13–23.

Solutions were obtained for two variable shear flows past a circular cylinder. The effects of shear on the displacement of the stagnation streamline was investigated. *Y. H. Kuo* (Peking).

Rosenblat, S.; and Woods, L. C. A method of cascade design for two-dimensional incompressible flow. Commonwealth of Australia. Dept. of Supply. Austral. Aero. Res. Comm. Rep. ACA-58 (1956), 30 pp.

This paper describes and exemplifies a method of designing cascades of aerofoils which in steady two-dimensional inviscid incompressible flow would have a given distribution of velocity over their surfaces. The method has the following advantages over that of the reviewer [Aero. Res. Council, Rep. and Memo. no. **2104** (1945); MR **8**, 610], to which it is in other respects similar. (i) The velocity is prescribed directly (instead of "semi-inversely") as a function of perimeter distance. (ii) Apart from the basic theory the method is essentially numerical, so that the velocity distribution does not have to be a function analytically simple enough for its conjugate to be found analytically. (iii) It allows for a non-zero trailing-edge angle. (iv) It is quicker and more flexible and, for example, can be used to calculate the modifications to existing cascades needed to improve their (experimentally determined) velocity distributions. *M. J. Lighthill.*

Woods, L. C. On harmonic functions satisfying a mixed boundary condition with an application to the flow past a porous wall. Appl. Sci. Res. A. **6** (1957), 351–364.

The author discusses the problem of determining the steady inviscid incompressible two-dimensional flow of a gas past a porous surface. As is known, the magnitude of the velocity vector and the angle which this vector forms with the x-axis are the real and imaginary parts of a function of a complex variable. The boundary values lead to a non-linear relation between these two parts. By linearizing this relation, the author is led to two Dirichlet problems, which are solved by an interesting use of integration in the complex plane. [For other solutions by Carleman and Gakhov, reference is made to Muskhelišvili, Singular integral equations, OGIZ, Moscow-Leningrad, 1946; MR **8**, 586; **15**, 434.] Finally, two applications of the theory are considered: (1) flow through a channel with porous walls; (2) an airfoil in a slotted tunnel. The results of the second example lead the author to suggest that the "porosity" of a wall should be determined from one of his parameters. *N. Coburn* (Ann Arbor, Mich.).

McDevitt, John B. The linearized subsonic flow about symmetrical nonlifting wing-body combinations. NACA Tech. Note no. **3964** (1957), 67 pp.

Pressure distributions, which are valid both in the near and far fields, are determined for non-lifting wing-body combinations in steady, subsonic flow. For this purpose linearized, rather than slender body, theory is used. In the case of a wing alone, the merits of both a surface distribution of sources and a distribution of line sources are considered. It is shown that the latter are simpler to use even in the case of tapered wings. For axially symmetric bodies alone an axial distribution of simple sources is sufficient to satisfy the boundary conditions; but, if the bodies are not axially symmetric, then multipoles are required. Finally, a treatment of wing-body combinations is given and some theoretically predicted pressures (for points near the wing-body junction) are compared with some experimental results obtained in the Ames 14 foot transonic wind tunnel. The comparisons are given for wings of two aspect ratios in combination with Sears-Haack bodies of revolution and bodies indented according to the transonic area rule. The two sets of results agree reasonably well. *G. N. Lance* (Southampton).

★**Bartels, R. C. F.; and Downing, A. C., Jr. On surface waves generated by travelling disturbances with circular symmetry.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 607–615. American Society of Mechanical Engineers, New York, 1955. $9.00.

This is a new treatment of a classic problem in the linearized theory of water waves, the determination of the shape of the surface wave resulting when a prescribed pressure distribution moves over the surface with constant velocity. Instead of finding the velocity potential first, the authors formulate an integro-differential equation for the function describing the surface. This is solved by a combined use of Fourier and Laplace transforms. Special attention is given to the case of pressure distributions with circular symmetry and to the case, treated by Kelvin, of a point disturbance. For the latter case it is shown that at least for a certain class of distributions the limiting procedure leads always to the same result. The authors point out, but do not elaborate, a discrepancy between their results for the surface along the path behind a point disturbance and those of Peters [Comm. Pure Appl. Math. **2** (1949), 123–148; MR **11**, 480].

J. V. Wehausen (Berkeley, Calif.).

Chabert d'Hières, Gabriel. Sur les équations approchées du clapotis parfait monochromatique. C. R. Acad. Sci. Paris **244** (1957), 2474–2476.

The author treats plane standing waves of finite amplitude under gravity. The depth is assumed constant, viscosity and surface tension are neglected. A formal scheme in terms of a Lagrangian description of the motion is set up; it is supposed that the displacements are expanded as power series in a steepness parameter. No explicit formulae are given. *F. Ursell.*

Daubert, André. Sur les équations approchées des ondes permanentes et périodiques de gravité. C. R. Acad. Sci. Paris **244** (1957), 2472–2474.

The depth is assumed constant, viscosity and surface tension are neglected but vorticity is included. The author takes the horizontal distance and the stream function as independent variables and sets up a formal scheme for calculating the vertical displacement as a power series in a steepness parameter. No explicit formulae are given.
F. Ursell (Cambridge, Mass.).

★**Schmieden, C.; und Müller, K. H. Die Strömung einer Quellstrecke im Halbraum — eine strenge Lösung der Navier-Stokes-Gleichungen.** Forschungsberichte des Wirtschafts- und Verkehrsministeriums, Nordrhein-Westfalen, Nr. 256. Westdeutscher Verlag, Köln und Opladen, 1956. 29 pp. DM 8.80.

The authors consider viscous flow from a source along the z-axis, with the x, y plane as a solid boundary. The boundary condition at infinity is obtained by requiring that the solution approach the inviscid flow as $\nu\to 0$. The introduction of the Stokes stream function enables the problem to be set up as a boundary value problem for a single non-linear fourth-order partial differential equation. By separation of variables this is reduced to a fourth order non-linear ordinary differential equation (with $W\equiv\cos\theta$ as independent variable) which can be integrated directly three times. The constants of integration depend on an unknown parameter, which can be chosen as the second derivative ϕ_0'' of the stream function, evaluated at the solid boundary ($W=0$ or $\theta=90^0$). By a suitable change of variable this equation can be made linear, leading to an exact solution of the problem.

Two cases must be distinguished: for a source of any strength, or for a sink of strength $<2\nu$ ($q<1$), there are infinitely many solutions (depending on the parameter ϕ_0''). For a sink of strength $\geqq 2\nu$ ($q\geqq 1$), on the other hand, ϕ_0'' is determined from an eigenvalue problem, and only one solution is obtained. The difference arises from the role of q in the differential equation

$$v(1-v)\frac{d^2u}{dv^2}+\frac{du}{dv}[q-(p+q)v]-\frac{(uq)}{2}[p+2-q-\phi_0'']=0.$$

The problem is solved in spherical co-ordinates. The apparently natural cylindrical co-ordinates lead to a differential equation which eludes efforts to integrate it. The authors compare their treatment with that of van Wijngaarden [Nederl. Akad. Wetensch., Proc. **45** (1942), 269–275; MR **5**, 247]. *R. B. Davis* (Syracuse, N.Y.).

Agrawal, H. L. A new exact solution of the equations of viscous motion with axial symmetry. Quart. J. Mech. Appl. Math. **10** (1957), 42–44.

The Stokes's stream-function in this exact solution is in the form $r^4 f(\cos\theta)$, where r, θ are spherical polar coordinates. The function f can be simply expressed in terms of Legendre functions, and in a particular case is proportional to $\sin^2 2\theta$. *W. R. Dean* (London).

Fujikawa, Hiroomi. Expansion formulae for the forces acting on two equal circular cylinders placed in a uniform stream at low values of Reynolds number. J. Phys. Soc. Japan **12** (1957), 423–430.

A formula is found from Oseen's approximate equations for two-dimensional flow for the forces on the cylinder; it gives the most important term for a small Reynolds number expressed as a series in a/h up to the order $(a/h)^4$. The forces are computed for some values of a/h, including the value 1 when the cylinders are in contact (a is the radius of either cylinder, and $2h$ the distance between their axes). This is a continuation of an earlier paper [same J. **11** (1956), 558–569; MR **17**, 1089] in which the author found a formula to order $(a/h)^2$ expressed as a series in the Reynolds number. *W. R. Dean.*

Constantinescu, V. N. L'écoulement laminaire des gaz en couches minces. Com. Acad. R. P. Romîne **6** (1956), 281–284. (Romanian. Russian and French summaries)

The author's investigation is based on the lubrication equation as obtained by N. Tipei [same Com. **4** (1954), 501–507, 699–704; MR **17**, 424]. In the case at hand, the distance between the two surfaces through which the flow takes place is considered very small. Exact expressions are obtained for two-dimensional steady flow (including variation of viscosity with temperature). In the case of three dimensions, when the distance is constant, the equations reduce to the two-dimensional Laplace equation. This case is, however, not treated explicitly.
K. Bhagwandin (Oslo).

Oroveanu, T.; et Ionescu, P. Détermination des pertes de fluide à travers l'espace compris entre le piston et le cylindre d'une pompe. Com. Acad. R. P. Romîne **6** (1956), 871–876. (Romanian. Russian and French summaries)

The author's object is to determine the loss of fluid across a region bounded by the piston and the cylinder of a pump. The problem is reduced to the determination of the laminar motion in the region bounded by two coaxial cylinders. The velocity of the interior cylinder varies with time.

The authors obtain explicit expressions for the velocity distribution and the discharge. These expressions contain the Bessel functions $J_0(z)$ and $Y_0(z)$ under the integral sign.

It should be noted, however, that a vast amount of computational work has to be carried out before any physical verification of these solutions can be established.
K. Bhagwandin (Oslo).

Proudman, Ian; and Pearson, J. R. A. Expansions at small Reynolds numbers for the flow past a sphere and a circular cylinder. J. Fluid Mech. **2** (1957), 237–262.

This treatment of the title problem by matching "Stokes expansions" of the stream function ψ, of the form

$$\psi=\sum f_n(R)\psi_n(r,\theta), \tag{1}$$

where R is Reynolds number, r, θ are appropriate polar coordinates and (1) satisfies the boundary condition at the solid surface, with "Oseen expansions" of the form

$$\psi=\sum F_n(R)\Psi_n(Rr,\theta), \tag{2}$$

where (2) satisfies the boundary condition at infinity, is closely parallel to the treatment by Lagerstrom and Cole [J. Rational Mech. Anal. **4** (1955), 817–882; MR **17**, 1021] but goes into rather more detail. *M. J. Lighthill.*

Cohen, Hirsh; and Gilbert, Robert. Two-dimensional, steady, cavity flow about slender bodies in channels of finite breadth. J. Appl. Mech. **24** (1957), 170–176.

The drag coefficient of a plate in a channel has been calculated elsewhere [G. Birkhoff, M. Plesset, and N. Simmons, Quart. Appl. Math. **8** (1950), 151–168; **9** (1952), 413–421; MR **12**, 297; **13**, 395], as a function of the cavitation number Q. Using the linearized approximation of M. Tulin [Taylor Model Basin Rep. **834** (1953); MR **15**, 261], the authors make similar calculations for wedges. Cavity lengths and blockage ratios are also computed.

G. Birkhoff (Cambridge, Mass.).

Görtler, Henry. On the calculation of steady laminar boundary layer flows with continuous suction. J. Math. Mech. **6** (1957), 323–340.

This well written paper is an extension of the author's paper [same J. **6** (1957), 1–66; MR **18**, 843], which presented a series method for the calculation of the steady, incompressible, laminar boundary layer along impermeable walls, to the case where suction or blowing out is permitted. It is shown that the coefficients in the series expansions can be represented as certain linear combinations of universal functions for reasonably arbitrary distributions of outer velocities $V(x)$ and suction velocities $v_0(x)$. Although the universal functions for $v_0(x)=0$ are still used, the number of universal functions required for this problem is considerably greater. The author states that it is planned to evaluate the necessary additional universal functions for the cases of flows over bodies with forward cuspidal points including flat plates with arbitrary pressure distributions and bodies with rounded noses. *R. C. DiPrima* (Troy, N.Y.).

Coleman, W. S. Comments on some recent calculations relating to the laminar boundary layer with discontinuously distributed suction. J. Roy. Aero. Soc. **61** (1957), 359–361.

The thickening laminar boundary layer (Reynold's number approaching transitions Reynold's number) can be thinned and stabilized by the placement of spanwise slots. It has been pointed out however that surface roughness limits the intensity of suction which can be applied at a suction slot. [Lachmann, Aero. Engrg. Rev. **13** (1954), no. 8, 37–51; J. Roy. Aero. Soc. **59** (1955), 163–198].

In this paper it is assumed that the fluid removed at the slot is equivalent to cutting off the lower portion of the boundary layer profile at the upstream edge of the slot. By the use of the integrated momentum equation it is shown (under the assumption that the slot width is small and the upstream and downstream velocities of the slot at the edge of the boundary layer are nearly equal) that the change in the momentum across the slot can be attributed solely to the withdrawal of the fluid at the slot. Computations are carried out for the case of a linearly varying velocity at the edge of the boundary layer; and the amount of fluid removed at a slot is plotted as a function of the ratio of the momentum thickness downstream to upstream of the slot.

Finally the author mentions that the conditions described in the first paragraph are appropriate only if there is no tendency of the flow to separate; and in some computed cases separation did tend to occur even though the Reynold's number was between a lower bound (determined by the roughness condition) and a transition Reynold's number. *R. C. Di Prima* (Troy, N.Y.).

Lachmann, G. V. Some observations on Dr. Coleman's comments. J. Roy. Aero. Soc. **61** (1957), 361.

[See previous review]. The author points out that in free flight tests with the spanwise suction which he adopted it was never observed that separation took place, and transition could be prevented by suitable adjustment of suction. However with wide spacing of the suction strips separation might possibly occur. It is suggested that for boundary layer control closely spaced suction strips with weak suction at each strip is preferable to concentrating stronger suction at fewer discrete slots. *R. C. Di Prima.*

Chambré, Paul L. The laminar boundary layer with distributed heat sources or sinks. Appl. Sci. Res. A. **6** (1957), 393–401.

The incompressible laminar boundary layer with distributed heat sources or sinks is studied. Although the problem can be treated when Q (the heat addition or withdrawal) is an arbitrary function of position, this analysis is restricted to the case where Q is uniformly distributed normal to the surface and varies in a prescribed fashion along the direction of flow. For the distribution chosen here the solution of the energy equation with Prandtl number $=1$ can be related to the solutions of a class of two-dimensional free stream flows with the geometric feature that the entire streamline pattern can be obtained by a translation of any particular streamline parallel to the leading edge.

Temperature profiles at two positions along the plate are given for flow past a plate at constant temperature, and the sources or sinks are distributed uniformly in the direction of the flow. *R. C. Di Prima* (Troy, N.Y.).

Curle, N. On hydrodynamic stability in unlimited fields of viscous flow. Proc. Roy. Soc. London. Ser. A. **238** (1957), 489–501.

The author develops an idea initiated by McKoen [Aero. Res. Council, Current Papers no. **303** (1956)] to calculate the stability characteristics of jets and wakes. In particular, the case of a two-dimensional jet issuing from a narrow slit is considered. A lower branch of the stability curve is obtained, and the minimum critical Reynolds number, (based on the quantities a and b in the velocity distribution $u=a\operatorname{sech}^2 by$), is found to be about 5.5. The main approximation used is the assumption that the fourth derivative is significent only near to the singular layer. It appears to this reviewer that this is not an accurate approximation at Reynolds numbers as low as 5.5 and associated wave numbers of the order of 0.2.

C. C. Lin (Cambridge, Mass.).

Hartley, R. V. L. Rotational waves in a turbulent liquid. J. Acoust. Soc. Amer. **29** (1957), 195–196.

Equations are derived for the propagation in a liquid, in a state of fine scale turbulence, of a type of so-called transverse wave in which the elasticity is associated with the rotation of an element rather than with its shearing as in a solid. This elasticity is proportional to the density of the kinetic energy of turbulent motion. The equations are identical with those for Kelvin's gyrostatic ether model and, for small displacements, are the analog of those of Maxwell for free space. (Author's abstract.)

A. H. Taub (Urbana, Ill.).

Kraichnan, Robert H. Pressure fluctuations in turbulent flow over a flat plate. J. Acoust. Soc. Amer. **28** (1956), 378–390.

This paper is aimed at predicting the character of the pressure fluctuations at the wall in a turbulent boundary layer. These fluctuations are responsible for the sound transmitted through the wall. An experimental determination of these pressure fluctuations has been given by W. W. Willmarth [same J. **28** (1956), 1048–1053]. The theoretical paper predicts from similarity arguments that the root-mean-square pressure fluctuation at the wall, divided by the main-stream dynamic pressure, should be independent of Mach number M and Reynolds number R, and the experiments showed that this was the case for $0.2<M<0.8$ and $1.5\times10^6<R<20\times10^6$. However, the experimental value for this ratio was 0.0035, a value closely comparable with the skin-friction coefficient, while the actual theoretical prediction is six times as much.

The theory is based on the assumption of a turbulence structure which differs from homogeneous isotropic turbulence only in being "symmetrized" with respect to the plane wall. The pressure is then obtained as a solution of Poisson's equation in unbounded space with a right-hand side depending bilinearly on the assumed turbulence [whose correlation-integral scale is taken from measurements by Laufer, NACA no. Rep. **1053** (1951)] and on an assume exponential distribution of mean shear which is fitted to the observed distribution in a region just outside the viscous sub-layer. The root mean square value so obtained is then reduced by a factor of 2 to allow for the finiteness of the actual region of mean shear. Predictions of the pressure spectrum and correlation functions on the wall are also made, and the effects of wall vibration discussed. *M. J. Lighthill* (Manchester).

Walters, T. S. Diffusion from an infinite line source lying perpendicular to the mean wind velocity of a turbulent flow. Quart. J. Mech. Appl. Math. **10** (1957), 214–219.

Une source linéaire indéfinie émet dans l'atmosphère turbulente une matière qui diffuse. Elle est placée perpendiculaire au vent, à la hauteur h au dessus du sol. La vitesse $U(z)$ du vent et le coefficient de diffusion K_z à la hauteur z sont supposés, tels que

$$U(z)=U_1\frac{z^m}{z_1{}^m},\quad K^z=AZ^\alpha.$$

Tenant compte des conditions aux limites, utilisant des changements de variables et une transformation de Hankel, l'auteur résout complètement l'équation de la diffusion. Il en déduit la distance du point au sol où la concentration est maximum, et la valeur de la concentration en ce point. *J. Bass* (Paris).

★**Carafoli, Elie. High-speed aerodynamics (compressible flow).** Editura Tehnică, Bucharest, **1956**. 710 pp. (2 plates).

In his book on "High-Speed Aerodynamics," Carafoli starts with introductory chapters on vector analysis and thermodynamics, and proceeds to an exhaustive treatment of the subject, which, in view of today's state of this field, is an almost heroic undertaking. It is inevitable that the author's personal interests and field of research will show in the emphasis put on the different chapters. Thus, in the chapter on subsonic flow the author does not content himself with reducing several problems to the incompressible case, but proceeds to solve and discuss them. Also, the chapters on linearized supersonic wing theory and on canonical flow abound with a wealth of explicitly treated examples, although (as in many other presentations of this subject) the author does not intend a comparably broad coverage of the different methods available in this field. Besides these remarkable chapters, the non-linearized theories are treated in a more conventional way, while second-order theories are not covered except for Busemann's case. A chapter on unsteady linearized flow is appended, while the theory of strong unsteady disturbances is not elaborated. Due to its particular emphasis, the book will be very valuable for specialists of wing theory, but it can be recommended also for the zealous student, who will find a book going beyond most textbooks, without the specialization of the monograph and the bulk of the handbook. *N. Rott.*

Finn, R.; and Gilbarg, D. Asymptotic behavior and uniqueness of plane subsonic flows. Comm. PureA ppl. Math. **10** (1957), 23–63.

This is a rigorous mathematical treatment of the theory of two-dimensional irrotational flows in which a functional relationship between density and speed exists and satisfies certain conditions. The main theorem is that if a flow is subsonic for $r\geqq r_0$ then it is uniformly subsonic for $r\geqq r_0$ (the Mach number has an upper bound <1), and the velocity vector tends to a constant subsonic value (which without loss of generality may be taken as $(u_0, 0)$) as $r\to\infty$; furthermore, the velocity potential ϕ possesses a convergent expansion of the form

$$\phi=u_0r\cos\theta+\frac{\Sigma}{2\pi\beta}\log r+\frac{\Gamma}{2\pi}\tan^{-1}(\beta\tan\theta)+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{c_m(\theta)}{r^n}\left(\frac{\log r}{r}\right)^n$$

if the "source strength" Σ or the "circulation" Γ is non-zero, and of the simplier form

$$\phi=u_0r\cos\theta+\sum_{n=0}^{\infty}\frac{c_n(\theta)}{r^n}$$

if $\Sigma=\Gamma=0$; where the Mach number at infinity is $\sqrt{(1-\beta^2)}$, and $c_{nm}(\theta)$ and $c_n(\theta)$ are functions with period 2π.

The authors also show that the subsonic flow around an arbitrary aerofoil, subject to certain conditions as to shape, is uniquely determined by the Kutta-Joukowski condition, that $\Sigma=0$ for this flow, that the drag is zero, the lift related to Γ, and the moment to $c_{01}(\theta)$ or $c_1(\theta)$. Much of this had been proved previously but comes out more readily by their methods.

They prove also that no partly supersonic flow past the given aerofoil can exist with the same circulation and velocity at infinity, and they investigate details of the flow near the trailing edge. *M. J. Lighthill.*

★**Chen, Y. W. Discontinuity and representations of minimal surface solutions.** Proceedings of the conference on differential equations (dedicated to A. Weinstein), pp. 115–138. University of Maryland Book Store, College Park, Md., 1956.

The author proves the existence of a solution $z(x, y)$ of the minimal surface equation

$$(1)\quad\left(\frac{z_x}{\sqrt{(1+z_x{}^2+z_y{}^2)}}\right)_x+\left(\frac{z_y}{\sqrt{(1+z_x{}^2+z_y{}^2)}}\right)_y=0,$$

which is defined exterior to a closed convex curve C, such that (i) ∇z tends to a prescribed finite limit at infinity, and (ii) $\partial z/\partial n=0$ on C in a generalized sense. The curve C is assumed to possess at most a finite number of protruding corners. A motivation for the problem can be found in the fact that solutions of (1) can be viewed as local approximations to "subsonic" solutions of the equations

$$(2)\quad (\rho z_x)_x+(\rho z_y)_y=0,\ \rho=[1-\frac{\gamma-1}{4}(z_x^2+z_y^2)]^{1/(\gamma-1)}(\gamma>1),$$

which describes the flow of an ideal compressible fluid. For a smooth contour C, the author's result is contained in earlier works of Bers [Trans. Amer. Math. Soc. **70** (1951), 465–491; MR **13**, 245; Comm. Pure Appl. Math. **7** (1954), 441–504; MR **16**, 417] and of Shiffman [J. Rational Mech. Anal. **1** (1952), 605–652; MR **14**, 510]. The author's method, however, differs from those of Bers and of Schiffman, and is of interest in itself. The problem is first solved in terms of uniformizing parametric coordinates (u, v) by means of an established variational procedure. The chief new difficulty is to show that the Jacobian $\partial(x, y)/\partial(u, v)$ does not vanish for this solution. This is done by a careful study of the behavior of the "Tchaplygin function"

$$\zeta=\frac{z_x-iz_y}{1+\sqrt{(1+z_x^2+z_y^2)}}$$

which is an analytic function of $u+iv$ for any solution, and is obviously bounded in magnitude. {Reviewer's remark: the special properties of this function have been used in several recent studies of (1) and of related equations.}

In the case that C is a convex polygon, the author shows that $\partial z/\partial n$ exists and vanishes at all points of C except at the vertices, where the solution surface necessarily contains a vertical segment. An interesting contrast appears between the behavior of solutions of (1) and those of (2). In a work D. Gilbarg and the reviewer [see the paper reviewed above] it is shown that in general there can be no single-valued "subsonic" solution of (2) satisfying (i) and (ii) when C contains a protruding corner.

As a corollary of his method, the author shows that the corresponding problem, in which (ii) is replaced by (ii') $z=0$ on C, admits no solution for any polygonal contour C. *R. Finn* (Pasadena, Calif.).

Poritsky, H.; and Powell, R. A. Point source and point vortex in the hodograph plane. J. Appl. Mech. **24** (1957), 183–190.

The solutions discussed are particular solutions of the usual hodograph equations of compressible flow, with the property that the streamlines in the hodograph plane either (i) diverge in all directions from a certain point not the origin, or (ii) are a set of closed curves with the point limit. Some indication is given of the use to which these solutions might be put in constructing possible flows about aerofoils, and there is a fairly full discussion of their relation to exact and approximate solutions which have been given earlier. *M. J. Lighthill.*

Stewart, R. W. Irrotational motion associated with free turbulent flows. J. Fluid Mech. **1** (1956), 593–606.

The irrotational motions induced by turbulent jets and wakes in the surrounding fluid are examined, both the mean flow consequent on entrainment at the boundary of the turbulent flow and the flow fluctuations caused by fluctuations at this boundary. It is shown that inhomogeneity in the general direction of flow does not lead to predictions of fluctuation intensities that are substantially different from those of Phillips [Proc. Cambridge Philos. Soc. **51** (1955), 220–229; MR **16**, 640] who neglected inhomogeneity, but that another mode of fluctuation parallel to the external mean flow streamlines is possible. Equations are given specifying the potential mean flow in the neighborhood of circular and two-dimensional turbulent jets. There is also a useful discussion of the mean flow velocity between bulges of turbulent fluid and of the effect of the apparent Reynolds stresses set up in the non-turbulent fluid. *A. A. Townsend* (Cambridge, England).

Wheelon, Albert D. Spectrum of turbulent fluctuations produced by convective mixing of gradients. Phys. Rev. (2) **105** (1957), 1706–1710.

A turbulent motion convects intensive properties of a fluid and tends to increase gradients of these properties. The author considers this process for a scalar property, specifically electron density, which has no effect on the motion. Supposing a constant established gradient of mean density, equations are derived for the density fluctuations and for their Fourier transform. By neglecting the effect of the turbulent motion on the density fluctuations, the fluctuation spectrum is obtained in terms of the velocity spectrum. This effect is dominant in the "inertial" range of wave-numbers, and dimensional reasoning is used to show that the spectrum varies as the square of the mean gradient and inversely as the cube of the wave-number. A general form for the spectrum over the whole "equilibrium" range of wave-numbers is derived by analogy with the Heisenberg "eddy viscosity" formulation of the transfer function describing energy transfer between different wave-numbers. It must be pointed out that the "eddy diffusivity" used here is essentially dissipative and not transferent as in Heisenberg's original hypothesis. *A. A. Townsend.*

Chuang, Feng-kan. On the decay of turbulence. Acta Sci. Sinica **2** (1953), 187–200.

This paper considers the decay of isotropic turbulence by assuming (a) similarity of all aspects of the motion not directly involved in viscous dissipation of energy and (b) that this part of the motion is specified uniquely by the rate of energy dissipation per unit mass and by its time derivative. Using Loitsiansky's invariant relation, a decay law is deduced in which the energy decays asymptotically as the $-10/7$ power of the time. In the second part of the paper, a similar analysis is applied to the approach to the final period of decay and it is shown that, to a first approximation, turbulent transfer between Fourier components of different wave-numbers is equivalent to an added viscosity. *A. A. Townsend* (Cambridge, England).

Taylor, Geoffrey; and Saffman, P. G. Effects of compressibility at low Reynolds number. J. Aero. Sci. **24** (1957), 553–562.

Reiner [Researches on the physics of air viscosity, Tech. Rep., Technion Res. Develop. Foundation, Haifa, 1956] has observed positive pressures of up to half an atmosphere in the space between two parallel plates when they are very close together and one rotates. This he explains as a consequence of the non-Newtonian viscosity of air. This paper inquires whether this effect might occur as a result of inevitable mechanical imperfections in the apparatus. Two imperfections are considered in detail,

(A) slight departures from parallelism, (B) longitudinal vibration of the rotor axis, assuming the Reynolds number to be so small that inertial forces are negligible. Only if the compressibility of the air is considered do these and other imperfections have the effect of raising the mean pressure at all radii including the centre. Analytical solutions are given for small displacements from parallel rotation, and a relaxation solution by G. Vaisey is given for a large deviation from parallelism. For imperfections of the expected magnitude, mean pressures of the same order as those observed are predicted. The existence of a similar effect in air-lubricated bearings is also discussed, and it is shown that a positive pressure is expected whenever rough or wavy surfaces slide over one another.

A. A. Townsend (Cambridge, England).

Jabotinsky, E. A tentative explanation of the Weissenberg and Reiner effects by postulating the existence of gyroscopic phenomena in viscous flow. Bull. Res. Council Israel. Sect. A. **6** (1956), 65–76.

The so-called Weissenberg effect in "visco-elastic" fluids which in a rotating fluid manifests itself by a "centripetal pump" effect has been attributed by Reiner to second order cross-effects due to the appearance in both the elastic and the viscous components of the constitutive equation of the fluid of second-order terms (square of the strain- and strain-rate deviations respectively). In the present paper the author attempts to derive this same effect from the Navier-Stokes equation by assuming the existence of gyroscopic effects in the classical viscous fluid, leading possibly to turbulent flow. The radial pressure distribution resulting from this assumption is established, providing a basis for its future experimental verification. *A. M. Freudenthal* (New York, N.Y.).

Bonder, Julian. Ondes simples dans les écoulements compressibles plans en régime non stationnaire. Arch. Mech. Stos. **8** (1956), 647–670.

Non-stationary, isentopic flows of a perfect gas in the plane are treated by seeking solutions of the equations of motion in the form of generalized simple waves. The sound speed a depends on a single variable (the density ρ or the pressure p). There are two relations between the sound speed and the flow components, $x=f_1(a)$ and $v=f_2(a)$, from which we have $du/da=2(k-1)^{-1}g(a)$ and $dv/da=2(k-1)^{-1}h(a)$. Hence, the equations of motion transform to the homogeneous linear system

$$\begin{aligned} a_t\ &+(u+ag)a_x+(v+ah)a_y=0,\\ ga_t&+(ug+a)a_x+vga_y\qquad\ \ =0,\\ ha_t&+uha_x\qquad\quad\ +(vh+a)a_y=0. \end{aligned}$$

The condition for a non-trivial solution is $g^2+h^2=1$. The system finally reduces to

$$a_t+(u+ag)a_x+(v+ah)a_y=0,$$
$$ha_x-ga_y=0,$$

for the single unknown function $a(t, x, y)$. This is a complete differential system with the general integrals

$$F_1[a, x-(u+ag)t, y-(v+ah)t]=0.$$
$$F_2[a, t, gx+hy]=0,$$

and the simultaneous general integral

$$F[a, gx+hy-(a+ug+vh)t]=0.$$

The method is applied to a detailed analysis of the non-stationary flow generated by the displacement of a dihedral through a gas from an initial condition of rest. Finally, the classical one-dimensional gas flow of Riemann and the supersonic stationary plane flow of Prandtl-Meyer are recovered as direct and particular consequences of the theory. *C. D. Calsoyas* (Livermore, Calif.).

Krajewski, Bohdan. Application of the variational method to the problem of axisymmetric rotational compressible flow. Arch. Mech. Stos. **9** (1957), 211–215. (Polish and Russian summaries)

The author outlines a method for determining the steady axial symmetric flow of a compressible inviscid duid with no body forces. The method consists of introflucing the stream function as a function of the radial distance and using the calculus of variations. Because of lack of clarity in specifying the independent variables, the reviewer was unable to follow the argument.

N. Coburn (Ann Arbor).

Săvulescu, St. Une méthode expéditive pour l'étude des caractéristiques de la couche limite. Com. Acad. R. P. Romîne **6** (1956), 877–883. (Romanian. Russian and French summaries)

The results of the author's investigation are accurately stated in the summary, and, therefore, it is reproduced below.

"L'auteur expose une méthode expéditive de détermination des caractéristiques de la couche limite, méthode procédant par approximations successives. Les équations de la couche limite sont transformées à l'aide de la variable adimensionelle $\eta=\psi/\psi_\delta$ où $\psi=\int_0^y \rho u\, dy$. La première approximation, du genre $u/u_e=\eta^n$, est introduite dans ces équations dont la structure est celle de l'équation de la diffusion, étant donné la forme de la tension de frottement et le transfert de chaleur qui dépendent des gradients de vitesse et, respectivement, de température. En supposant que n ne depend que du régime d'écoulement, laminaire ou turbulent, on déduit les caractéristique de la couche limite (frottement, transfert de chaleur, épaisseur δ) par une intégration double."

The expressions obtained by the author reduce in some some cases to well-known expressions. *K. Bhagwandin.*

Spreiter, John R.; and Alksne, Alberta Y. Thin airfoil theory based on approximate solution of the transonic flow equation. NACA Tech. Note no. **3970** (1957), 82 pp.

The authors seek approximate solutions, valid at transonic speeds, for the local perturbation velocity

$$u(x)=\varphi_x(x, 0)$$

on a thin airfoil of prescribed ordinate $Z(x)$ or, conversely, Z for prescribed u. This leads them to

$$\text{(A, B)}\qquad \lambda\varphi_{xx}+\varphi_{zz}=0,\quad (\varphi_z)_{z=0}=UZ'(x),$$
$$\text{(C)}\qquad \lambda=1-M^2-(\gamma+1)M^2U^{-1}\varphi_x,$$

where U and M denote free stream velocity and Mach number and γ specific heat ratio. They solve (A, B) for purely supersonic ($\lambda>0$) or purely subsonic ($\lambda<0$) flows by assuming λ to be constant and evaluate the local acceleration $u'(x;\lambda)$; then λ is substituted in u' with $\varphi_x=u$ and the resulting differential equation integrated to obtain $u(x)$, subject to $u=O(Z')$. For small $|M-1|$, they solve (A) on the assumption that

$$\lambda_1=(\gamma+1)M^2U^{-1}\varphi_{xx}$$

is constant, solve for $u(x;\lambda_1)$, let $\varphi_{xx}=u'(x)$ in λ_1, and integrate for $u(x)$ subject to its appropriate behavior at

the local sonic point. The supersonic result is shown to agree with Lighthill's transonic form of simple wave theory. [General theory of high speed aerodynamics, W. R. Sears, ed., Princeton, 1954, p. 387; MR **16**, 300], while the subsonic result compares satisfactorily with other approximations (e.g., Kármán-Tsien). The most impressive achievements, however, are for accelerating transonic flows ($\lambda_1>0$), where extensive comparisons of pressure distributions and total drag exhibit remarkable agreement with available (but far more complicated) theory and experiment; e.g., the authors' approximate determination of the sonic drag coefficients for a single wedge agrees with the results of exact (small disturbance) theory to better than 1%. Decelerating transonic flow ($\lambda_1<0$) is treated only briefly, for, as the authors point out, deceleration is not apt to be smooth (i.e., potential) in reality. Two examples of accelerating-decelerating flow, using the transonic ($\lambda_1>0$) approximation over the front of an airfoil and simple wave theory over its rear are presented. The ad hoc flavor of the author's approximations is somewhat alleviated by an appendix, in which these are shown to be superior to other possible approximations. The end impression is that the authors have brought about a tour de force of considerable, practical importance. *J. W. Miles* (Los Angeles, Calif.).

Adamskiĭ, V. B. Integration of a system of self-similar equations in the problem of a shock of short duration in a cold gas. Akust. Ž. 2 (1956), 3–9. (Russian)

The problem of propagation of a pressure pulse in a shock tube, half filled with quiescent gas in an initially uniform state, is considered, following earlier work by Zeldovich [Akust. Ž. **2** (1956), 1, 28–38; MR **18**, 620]. A similarity hypothesis is employed in which all physical quantities depend only on the single variable $\eta=m/M$, where $m=\int_{-\infty}^{x}\rho dx=\rho_0 y, M=\rho_0 X$; x, y are Eulerian and Lagrangian distances respectively, measured along the shock tube, X is the Eulerian coordinate at the shock, and ρ_0 is the initial gas density.

The pressure, velocity and density may then be written in the form

$$p=A\rho_0 M^{-n}f(\eta),\quad u=A^{-1/2}M^{-n/2}w(\eta),\quad \rho=\rho_0 q(\eta),$$

respectively. The Lagrangian equations of motion can then be reduced to ordinary differential equations for f, w and q. The solutions of these satisfying the shock wave conditions on $\eta=1$ are found and shown graphically. *M. Holt* (Providence, R.I.).

Chisnell, R. F. The motion of a shock wave in a channel, with applications to cylindrical and spherical shock waves. J. Fluid Mech. 2 (1957), 286–298.

L'auteur étudie le propagation des ondes de choc dans un tuyère de section lentement variable. Les équations du choc sont différentiées pour tenir compte de la variation de la section de la tuyère; par intégration une relation entre la section et l'intensité du choc est obtenue. Les résultats sont ensuite étendus aux ondes cylindriques et aux ondes sphériques; la comparaison de ces résultats avec les solutions exactes de Guderley [Luftfahrtforschung **19** (1942), 302–311; MR **5**, 19] et de Butler [Armament Res. Establishment Rep. no. **54/54** (1954)] fait apparaître une concordance. *H. Cabannes* (Marseille).

Whitham, G. B. A new approach to problems of shock dynamics. I. Two-dimensional problems. J. Fluid Mech. 2 (1957), 145–171.

This paper gives an approximate theory of the unsteady two-dimensional propagation of shock waves of arbitrary shape and strength into a uniform medium at rest. Previous theories have either made severe restrictions on shape (departures from the plane or circular shapes being required to be small) or on strength.

The author aims at evaluating the successive positions of the shock front. The orthogonal trajectories of this system of curves are called "rays". The motion of the part of the shock which lies between two rays can be regarded as in many mays similar to the motion of a shock wave along a channel of varying cross-sectional area A (proportional to the distance between the rays). Now, in the latter problem Chisnell has produced strong evidence [see the paper reviewed above] that the Mach number M of the shock is to a close approximation in a fixed functional relationship to the cross-sectional area A. The present author assumes that the local shock Mach number M remains in this fixed functional relationship to A in his more general problem. But since M and A are proportional respectively to the distance between successive shock-wave positions and successive rays, a differential relation between them exists, stating in effect that these systems of curves are orthogonal. After elimination of, say, A by means of the previous functional relationship, we are left with a second-order partial differential equation for the single unknown M.

Fortunately, this equation is of a very familiar type, that discussed by Riemann in his treatment of plane waves of finite amplitude in gases. Many solutions can therefore be derived speedily. These include problems of diffraction and of reflection of shock waves at corners of arbitrary angle, and problems of "shock-wave stability." Comparisons with known theoretical and experimental results where possible show remarkably good agreement. Discrepancies are substantial only for weak shocks, for which in any case other theoretical approaches are available.

An interesting result of the analogy with the Riemann theory is that reductions in shock strength are propagated along the shocks as disturbances dispersed over lengths of shock increasing with time (compare the "centred simple wave" in the Riemann theory), and that, contrariwise, disturbances involving increase of shock strength tend to run together. When such disturbances overtake each other they form discontinuities, which the author calls "shock-shocks" (compare ordinary shock formation in the Riemann theory). Physically, these are three-shock intersections and the present theory of them throws much light on the Mach reflexion problem and allied topics. *M. J. Lighthill* (Manchester).

Prosnak, Włodzimierz. Shock wave in a two-dimensional radial gas flow. Arch. Mech. Stos. 8 (1956), 617–645.

This treatment of two-dimensional "source" and "sink" flow in a viscous heat-conducting gas differs from earlier work by Sakurai [J. Phys. Soc. Japan. **4** (1949), 199–202; MR **12**, 454] and Levey [Quart. Appl. Math. **12** (1954), 25–48; MR **16**, 190] in taking the values 0 and ∞ for the Prandtl number instead of the value $0.75+R^{-1}$ (where the Reynolds number R is based on the mass flow per radian sector). Physical interpretation of the results is difficult, except that a part of the source flow is like flow in a radial supersonic effuser, with transition to diffuser behaviour through a shock wave. Trouble arises as usual with the treatment of shock wave structure in the case when $\sigma=0$ and the shock wave is strong. *M. J. Lighthill* (Manchester).

Potter, David S.; and Murphy, Stanley R. On wave propagation in a random inhomogeneous medium. J. Acoust. Soc. Amer. **29** (1957), 197–198.

Les auteurs corrigent les résultats obtenus par D. Mintzer, [même J. **25** (1953), 9227, 1107-1-92111; MR **15**, 481, 662] et obtiennent de meilleurs résultats que ceux obtenus par lui. *M. Kiveliovitch* (Paris).

Malyužinec, G. D. Radiation of sound by vibrating boundaries of an arbitrary wedge. I, II. Akust. Ž. **1** (1955), 144–164, 226–234. (Russian)

By employing the Sommerfeld integral representation for solutions of the wave equation in wedge-shaped regions, the author proceeds in paper I to solve exactly the acoustical problem of the radiation of sound from the vibrating boundaries of a wedge, where the velocities of vibration of the wedge faces are either constant (though different on the two boundaries) or vary sinusoidally. The solution is expressed in terms of a Sommerfeld integral whose integrand involves a special function $\eta_\Phi(z)$ which was studied by the author previously in connection with diffraction problems for a wedge with homogeneous boundary conditions. From the formal integral solution, a series representation is derived which converges rapidly for small values of $k\rho$ (k=wavenumber in the medium, ρ=distance from wedge apex), and an asymptotic expression for $k\rho\gg 1$ is obtained which is useful for computation outside the geometric-optical transition regions.

Although the author has preferred to solve this problem via the interesting, but somewhat lengthy Sommerfeld integral technique, it should be pointed out that an alternative method employing the known acoustical Green's function for a perfectly rigid wedge leads to the solution in a more direct and straightforward manner. If the half planes at $\varphi=0$ and $\varphi=\alpha$ constitute the boundaries of a wedge-shaped region containing a medium with density w and sound velocity c, and the wedge face at $\varphi=0$ vibrates with velocity $v_0 \exp(-i\omega t)$, v_0 constant, while the boundary at $\varphi=\alpha$ remains rigid, then one can show readily via an integration of the acoustical Green's function [Felsen, Microwave Res. Inst., Polytech. Inst. Brooklyn, Rep. **R–613-57**, **PIB-541** (1957)]: that the radiated sound pressure p is given by:

$$p(\rho, \varphi)=wcv_0S(\rho, \varphi),$$

$$S(\rho, \varphi)=I(\varphi+\tfrac{1}{2}\pi, \alpha)-I(\varphi-\tfrac{1}{2}\pi, \alpha), \quad \alpha\geqq\varphi>\tfrac{1}{2}\pi.$$

The function $I(\delta, \alpha)$ has been investigated in detail by Oberhettinger [J. Math. Phys. **34** (1956), 245–255; MR **17**, 476] in connection with wedge diffraction problems, and its asymptotic expansion for $k\rho\gg 1$ is available for all values of φ, including the geometric-optical transition regions $\delta\to 0$ (a series representation for $I(\delta, \alpha)$, which converges rapidly for small $k\rho$, is also given). Similar solutions can be obtained for $\varphi<\frac{1}{2}\pi$, and also for $\alpha<\frac{1}{2}\pi$.

In paper II, the author employs the solution obtained in his first paper to compute the radiated sound intensity, and he shows that the edge of the wedge produces only a reactive effect, i.e., it does not influence the radiated intensity. *L. B. Felsen.*

Lysanov, Yu. P. On an approximate solution of the problem of scattering of sound waves on an uneven surface. Akust. Ž. **2** (1956), 182–187. (Russian)

An approximate solution of the problem of diffraction of a plane wave by a corrugated surface is obtained by first formulating the integral equation satisfied by pressure on the diffracting surface. An approximate integral equation is then obtained and solved when the slope of the corrugations is small. The author also estimates the field in the Fraunhofer and Fresnel regions when the corrugations extend over a finite area. *J. Shmoys.*

Krasil'nikov, Yu. I. Unsteady motion of visco-plastic liquid in a pipe. Prikl. Mat. Meh. **20** (1956), 655–660. (Russian)

The problem considered is that of a flow of a visco-plastic (incompressible) liquid in a straight cylindrical pipe. The fundamental general set of equations used is that derived by G. Genki [On slow stationary flows in plastic media, Theory of plasticity (Yu. N. Rabotnov, ed.), Gos. Izdat. Inostran. Lit., 1948] reduced to the two-dimensional case in cylindrical polar coordinates with only the longitudinal velocity component being preserved. The inertia terms are neglected so that the equation is linearized. The surface stress tensor consists of two parts; elastic and viscous stress tensor. The initial condition superimposed upon the velocity is that at $t=0$ it must be equal to the given function of the radius. The boundary condition is that the velocity is zero at the wall of the pipe. Author proposes a solution for the velocity in form of the sum of two series with, for the time being, unknown coefficients, one of which is a trigonometric series and another a Fourier-Bessel series, subject to the corresponding initial and boundary conditions. By means of suitable manipulations the author obtains workable equations for the coefficients of the series. As a particular case Krasil'nikov considers the flow with constant pressure difference and derives the final formulas for the velocity and flux at $t=0$ and $t=\infty$. The reader is warned that the symbol ρ is used in double meaning; density and radius. *M. Z. Krzywoblocki* (Urbana, Ill.).

Talwar, S. P.; and Abbi, S. S. On the change in shape of a gravitating fluid sphere in a uniform external electric field. Proc. Nat. Inst. Sci. India. Part A. **22** (1956), 7–12.

It is shown that a conducting incompressible fluid sphere in a uniform external electric field is not a configuration of equilibrium and that it will tend to become a prolate spheroid in the direction of the electric field. *S. Chandrasekhar* (Williams Bay, Wis.).

Sykes, John. The equilibrium of a self-gravitating rotating incompressible fluid spheroid with a magnetic field. Astrophys. J. **125** (1957), 615–621.

Considering a prevailing axially symmetric magnetic field together with axially symmetric rotational motions in an incompressible non-viscous, infinitely conducting fluid, Chandrasekhar [Astrophys. J. **124** (1956), 232–243, eq. 55; MR **18**, 86] has shown that the general solution of the equation of equilibrium is given by

$$(1) \qquad \Delta_5 P=-T\frac{d(\varpi^2 T)}{d(\varpi^2 P)}-\varpi^2 V\,\frac{dV}{d(\varpi^2 P)}+\Phi(\varpi^2 P),$$

where P and T are the scalars characterizing the poloidal and the toroidal components of the magnetic field and V is the corresponding scalar characterizing the rotational motion. In (1) ϖ denotes the distance from the axis of symmetry, Δ_5 is the five-dimensional axially symmetric Laplacian operator, Φ is an arbitrary function of the argument specified, $\varpi^2 T$ is a function of $\varpi^2 P$ and V is a function of $\varpi^2 P$. The case $V=0$, $\Phi=K$=constant and $T=\alpha P$ where α is a constant has been considered by

Prendergast [ibid. **123** (1956), 498–508]. In this paper the discussion is extended to include the case $V^2=\beta^2\varpi^2P+V_0^2$ where V_0 is a constant. The equation to be solved then becomes

$$(2)\qquad \Delta_5P+\alpha^2P+\tfrac{1}{2}\beta^2\varpi^2-K=0.$$

A solution of this equation is sought which satisfies all the relevant boundary conditions on an oblate spheroid

$$(3)\qquad \varpi^2/a^2+z^2/b^2=1\quad (b^2=a^2(1-e^2)).$$

It is shown that all the required boundary conditions can be satisfied consistently to order e^4. Thus while Prendergast's spherical configuration required α to be a root of $J_{5/2}(\alpha R)=0$ (where R is the radius of the sphere) the presence of rotation requires α to be determined by

$$J_{5/2}(x)=\frac{3}{14}\,e^2xJ_{3/2}(x)+O(e^4),$$

where $x=\alpha a$. *S. Chandrasekhar.*

Scheidegger, A. E. Correlation tensors in statistical hydrodynamics in porous media. Canad. J. Phys. 34 (1956), 692–698.

Here is an occupational hazard of contributors to Mathematical Reviews new to this reviewer. A long and ponderous paper discussing a field of knowledge that has been thoroughly understood for many years underlines the obvious to such an extent that the reviewer finds it possible to state what has been done in three sentences while observing that the mathematical machinery could have been justified only if a certain refinement had been made. The author then replies to the review in another long discursive paper similar to the first, which is sent for review to the original reviewer. One only hopes that the process is convergent.

The reviewer's specific objection to the first paper [J. Appl. Phys. **25** (1954), 994–1001; MR **16**, 190] was that in studying porous media by taking ensemble averages over a large number of different but macroscopically identical specimens of porous material one gets trivial results easy to foresee in advance by using (in the author's words) "the statistics of complete disorder initiated by Einstein in his paper on Brownian motion". The investigation, to merit the space used on it, should have taken into account that there must be "autocorrelation between successive movements of particles of fluid" when analyzed by such ensemble averages. The author now takes seven pages to say that he has been unable to bring this Lagrangian correlation to bear on the problem by using Eulerian-correlation techniques from the theory of homogeneous turbulence. *M. J. Lighthill.*

Wooding, R. A. Steady state free thermal convection of liquid in a saturated permeable medium. J. Fluid Mech. 2 (1957), 273–285.

L'auteur présente une solution approximative pour le problème du titre au moyen des méthodes de perturbation. Les valeurs empiriques sont utilisées pour la variation de densité. L'auteur emploie les expansions en series pour la variation de la température et pour la fonction de courant. Quatre coefficients de l'expansion sont obtenus. Ces fonctions satisfont les conditions aux limites (les cas homogène et non-homogène) de Dirichlet et de Neumann. Un exemple numérique de la méthode est aussi présenté. Les résultats paraissent vraisemblables.

K. Bhagwandin (Oslo).

Barenblatt, G. I. On self-similar solutions of the Cauchy problem for a nonlinear parabolic equation of unsteady filtration of a gas in a porous medium. Prikl. Mat. Meh. **20** (1956), 761–763. (Russian)

In the present note the author studies the Cauchy initial-value problem for the non-linear parabolic equation

$$\frac{\partial p}{\partial t}=a^2\frac{1}{r}\frac{\partial}{\partial r}\left(p\frac{\partial p^2}{\partial r}\right)$$

subject to the conditions $p(p,0)=\sigma r^\alpha$, $\sigma=\text{const}>0$, $\alpha=\text{const}>0$ and $(\partial p/\partial r)_{r=0}=0$, where p denotes the pressure of the non-stationary axial-symmetrical isothermal filtrating gas in a porous medium, t is the time, r denotes the distance from the axis of symmetry, and a is a prescribed constant. The author distinguishes the three following cases. (1) When $0<\alpha<2$ a unique solution exists for all values of t $(0\leq t<\infty)$. This solution can be obtained by means of the substitution $p(r,t)=\sigma r^\alpha F(\xi)$, $\xi=r(a^2\sigma t)^{-\frac{1}{4}(2-\alpha)}$. The function $F(\xi)$ satisfies the following differential equation

$$\xi^\alpha\left[\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{dF^2}{d\xi}\right)\right]+\frac{4\alpha}{\xi^{1-\alpha}}\frac{dF^2}{d\xi}+\frac{4\alpha^2}{\xi^{2-\alpha}}F^2+\frac{1}{2-\alpha}\,\xi\frac{dF}{d\xi}=0$$

and the conditions

$$F(\infty)=1,\quad\left[\frac{d}{d\xi}\,\xi^\alpha F(\xi)\right]_{\xi=0}=0.$$

An explicit solution to this problem is obtained. (2) For $\alpha=2$ the solution can be obtained in closed form, viz.,

$$p(r,t)=\sigma p^2/(1-16a^2\sigma t)\quad(0\leq t<1/16a^2\sigma).$$

(3) The solution of the problem when $\alpha>2$ follows closely the reasoning of the first case. The analysis shows that for $\alpha>2$ the solution has no physical meaning.

Finally, the author also conjectures that a similar analysis can be applied in the case of a polytropic law of filtration, i.e., when the pressure satisfies the equation

$$\frac{\partial P}{\partial t}=a^2\Delta P^k,\quad P=p^{1/(k-1)},$$

where $k\geq1$. However, it seems that the polytropic case will not yield solutions of the same type as treated above. *K. Bhagwandin* (Oslo).

See also: Chandrasekhar and Reid, p. 144; Wilson, p. 174; Brittin, p. 190; Goody, p. 213.

Optics, Electromagnetic Theory, Circuits

Tai, C. T. Theory of the cylindrical Luneberg lens excited by a line magnetic current. The Ohio State University Research Foundation, Columbus, Ohio, Rep. 678–3 (1956), iii+8 pp.

The report discusses the transverse-magnetic modes associated with a cylindrical Luneberg lens when excited by a magnetic line.

This work is parallel to the equations of the electric mode discussed by H. Jasik [Air Force Cambridge Research Center Rep. **TR-54-121** (1954)], with the exception that the differential equation of the radial distribution $K(r)$ of the dielectric constant contains a term proportional to dK/dr.

The author suggests to develop the solution with the

help of hypergeometric functions. However, he does not investigate the asymptotic behavior of these solutions, which may tie up with the solution obtained by geometrical or physical optics. *M. Herzberger.*

Schiske, P. Bahnen 3. Ordnung in Elektronenspiegeln. Optik **14** (1957), 34–45.

Two different approaches are presented for the treatment of electron mirrors taking into account 3rd order aberrations; standard perturbation calculations are inapplicable since they assume a small angle between the electron path and the optical axis. The first approach is based on the general method of Recknagel [Z. Physik **104** (1937), 381–394] equally applicable to electron lenses and mirrors, where time is used as the independent variable. The second one involves a generalization of the Seidel eiconal, which reduces to the Seidel eiconal in the absence of a reflection point. To facilitate comparison between the two approaches, the first treatment is put in the form of canonical perturbation calculation. The eiconal calculation is carried out in rectangular coordinates. It is possible, however, to use other coordinates which are more convenient in the case of a magnetic field.

J. E. Rosenthal (Passaic, N.J.).

Nisbet, A. Electromagnetic potentials in a heterogeneous non-conducting medium. Proc. Roy. Soc. London. Ser. A. **240** (1957), 375–381.

For electromagnetic fields in a stationary non-conducting medium (isotropic or anisotropic), whose dielectric constant and permeability are given point-functions, the general theory of representations in terms of scalar and vector potentials and of Hertzian potentials is developed. Differential equations relating these potentials to the sources of the field are obtained. Special cases are also treated of representations in terms of two scalars satisfying second-order differential equations. This paper contrasts with a recent alternative approach [Nisbet, same Proc. **231** (1955), 250–263; MR **18**, 700], in which the properties of the medium were introduced through the electric and magnetic polarizations rather than the dielectric constant and permeability, and which led to integro-differential equations. (From the author's summary.) *A. E. Heins* (Pittsburgh, Pa.).

Matthews, P. A.; and Cullen, A. L. A study of the field distribution at an axial focus of a square microwave lens. Proc. Inst. Elec. Engrs. C. **103** (1956), 449–456.

Die Verfasser erhalten die Verteilung des elektromagnetischen Feldes in der Nähe des Brennpunktes einer Mikrowellenlinse.

Die skalaren Approximationen werden dabei benutzt. Die experimentellen Ergebnisse erscheinen in guter Übereinstimmung mit der theoretischen Analyse, und die 180° Phasenverschiebung wird experimentell verifiziert.

Die Genauigkeit der durchgeführten Analyse ist nicht streng, wesentliche Beiträge [z.B. Luneberg, Mathematical theory of optics, Brown Univ., Providence, R.I., 1944; MR **6**, 107] sind nicht berücksichtigt.

K. Bhagwandin (Oslo).

★**Müller, Claus. Grundprobleme der mathematischen Theorie elektromagnetischer Schwingungen.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957. ix+344 pp. DM 52.80.

While this book is primarily a text dealing with the mathematical theory of electromagnetic vibrations, it departs from the usual run of text books in that it emphasizes not only the author's point of view but also his original work. It aims at a unified presentation of the subject matter. The first two chapters are devoted to the necessary mathematical tools, primarily the basic ideas of vector analysis and spherical and Bessel functions. One of the novel features of this treatment is the definition of the basic operations of vector analysis as limiting processes rather than in terms of the partial derivatives of the vector fields. This permits the use of such concepts as divergence and curl in cases where they cannot be defined in the usual way. It is shown under what special conditions the two sets of definitions coincide. It is unfortunate, however, that the notation ∇^* was introduced to describe the operator ∇ as defined by the new method since the asterisk is used in the text to denote real quantities. The following chapters are devoted to the Helmholtz vibrational equation, the electromagnetic vibrations in homogeneous and inhomogeneous space, boundary value problems, and the radiation characteristics. The solution of Maxwell's equations in homogeneous space is given by integral representations, which can be interpreted as a mathematical formulation of the Huygens principle. The treatment of linear transformations follows F. Riesz [Acta Math. **41** (1916), 71–98] but appears to be more complete (and complicated) than needed solely in the study of electromagnetic vibrations. The chapter on radiation characteristics contains previously unpublished results of the author.

In his introductory remarks, Müller states that the underlying purpose of the book is a general exposition of the subject showing its range and completeness rather than a treatment of the multiple problems arising in the mathematical theory of electromagnetic vibrations and of the techniques needed to handle them. This reviewer feels that an approach of this type has been badly needed and that the author has succeeded in his purpose.

J. E. Rosenthal (Passaic, N.J.).

Zubarev, D. N. A generalization of the method of auxiliary variables. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 489–492. (Russian)

In the description of the motion of electrons in a plasma the auxiliary variables introduced to describe the collective motion have hitherto been plane-wave functions. In the present work it is proposed to replace these by arbitrary orthogonal functions which can be chosen to suit the conditions of the problem. This method can be applied also to the collective motion of a nucleus.

N. Rosen (Haifa).

Denisov, N. G. On a singularity of the field of an electromagnetic wave propagated in an inhomogeneous plasma. Soviet Physics. JETP **4** (1957), 544–553.

The phenomenon under investigation is the propagation of electromagnetic waves in an inhomogeneous plano-stratified medium. The normal incidence is the simplest special case, and is treated in the event total reflection occurs by using the linear approximation of the dielectric constant $\varepsilon(z)$ in the neighbourhood of its zero. The case of oblique incidence is somewhat more complicated, and several investigators [Zhekulin, Ž. Eksper. Teoret. Fiz. **4** (1934), 76; Forsterling and Wüster, J. Atmos. Terrest. Phys. **2** (1951), 22–31] have studied the singularities which arise where the dielectric constant is zero. The present paper discusses the problem of the amplitude in a

growing field within the absorbing medium, and the physical nature of this singularity in a medium without absorption. An approximate calculation of the effect of plasma waves is given. *M. J. Moravcsik* (Ithaca, N.Y.).

Sitenko, A. G.; and Stepanov, K. N. On the oscillations of an electron plasma in a magnetic field. Soviet Physics. JETP **4** (1957), 512–520.

The oscillations of an electron plasma in a magnetic field is discussed, tieing in with work done by Akhiezer and Pargamanik [Uč. Zap. Har'kov. Gos. Univ. **27** (1948), 75] and by E. P. Gross [Phys. Rev. (2) **82** (1951), 232–242; MR **12**, 886]. The basis of the treatment is the kinetic theory, and the dispersion equations are solved and investigated. The indices of refraction are determined for the ordinary, the extraordinary, and the plasma waves, propagated at an angle θ with respect to the magnetic field. The plasma waves are shown to be highly damped for $\theta<\frac{1}{2}\pi$ if the frequencies are integral multiples of the gyromagnetic frequency. In fact, for $\theta=\frac{1}{2}\pi$ plasma waves of such frequencies cannot be propagated at all. Thus there will be gaps in the spectrum the width of which are computed in the paper. *M. J. Moravcsik.*

Delavault, Huguette. Sur la résolution des équations de Maxwell en coordonnées cylindriques au moyen de transformations de Laplace et de transformations finies de Fourier et de Hankel. C. R. Acad. Sci. Paris **244** (1957), 1146–1149.

The author determines the electromagnetic field in a cylindrical region filled with matter subject to initial and boundary conditions by effecting the following transformations on the field components in cylindrical coordinates: Laplace transformation in t and z; Fourier transformation in θ; Hankel transformation in r.
N. L. Balazs (Chicago, Ill.).

Schützer, Walter. On Bohm-Pines theory of plasma. An Acad. Brasil. Ci. **28** (1956), 419–422.

Die von D. Pines und D. Bohm für ein Plasma (dichtes Elektronengas, dessen Ladung durch eingebettete positive Ionen kompensiert ist) erhaltenen Resultate [Phys. Rev. (2) **85** (1952), 338–353] werden nach einem anderen Gedankengange hergeleitet. Den Ausgangspunkt der Betrachtungen bildet die Hamiltonsche Funktion eines Elektrons

$$(1)\qquad H=\frac{1}{2m}\mathbf{p}^2+V_0+4\pi e^2\sum_k k^{-2}\int e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}f d\mathbf{x}'d\mathbf{p}$$

wo $f(\mathbf{x}\mathbf{p}t)$ die Dichtefunktion ist. Das erste Glied auf der rechten Seite von (1) ist die kinetische Energie, das zweite beschreibt den von den positiven Ionen herrührenden Potentialtopf und das dritte Glied enthält endlich die kollektive Coulombsche Wechselwirkung mit den übrigen Elektronen. Danach wird die Liouvillesche Differentialgleichung

$$(2)\qquad \frac{\partial f}{\partial t}+[f, H]=0$$

benutzt und nach einigen weiteren Rechnungen erhält der Verfasser die Gleichungen von Bohm und Pines. Der Vorteil dieser neuen Methode ist, nach dem Verfasser, dass man auf diesem Wege erstens auch ein endliches Plasma behandeln kann und dass diese Methode wenigstens ein qualitatives Bild für die auftretenden Dämpfungserscheinungen liefert. *T. Neugebauer.*

Ritchie, R. H. Plasma losses by fast electrons in thin films. Phys. Rev. (2) **106** (1957), 874–881.

Zur Erklärung der bei dem Durchgang von Elektronen durch dünne Metallfolien beobachteten charakteristischen Energieverlusten haben D. Pines und D. Bohm [Phys. Rev. (2) **85** (1952), 338–353] die Hypothese aufgestellt, dass diese Energieverluste von der Anregung der sogenannten Plasmaschwingungen in der Gesamtheit der Leitungselektronen herrühren. Eine andere Erklärung für diese Erscheinung wäre, dass die von den Übergangen der Leitungselektronen zwischen zwei Energiebänder im Metall herrühren. In der vorliegenden Arbeit wird nach der ersteren Auffassung gerechnet. Die Wechselwirkung des einfallenden Elektrons mit den Leitungselektronen wird dadurch berücksichtigt, dass die Gesamtheit der Leitungselektronen im Metall mit Hilfe einer Dielektrizitätskonstante beschrieben wird, die jedoch selbstverständlich keine Konstante ist. J. Lindhard [Danske Vid. Selsk. Mat.-Fys. Medd. **28** (1954), no. 8; MR **16**, 204] und J. Hubbard [Proc. Phys. Soc. Sect. A. **68** (1955), 976–986] haben diese Dielektrizitätskonstante mit Hilfe der quantenmechanischen Störungstheorie berechnet und die vorliegende Arbeit ist eigentlich eine Erweiterung und Präzisierung dieser Theorie. Zuerst wird der Fall des unendlich ausgedehnten Plasmas besprochen und dabei wird gezeigt, dass bei kleinen Ablenkungswinkeln das kollektive Verhalten der Elektronen die Ablenkungserscheinungen verursacht, für grössere Ablenkungswinkel oder sehr kleine Elektronendichten werden jedoch die individuellen Wechselwirkungen mit den Leitungselektronen massgebend. Bei der Besprechung des eigentlichen Problems einer dünnen Folie entsteht eine Schwierigkeit dadurch, dass die quantenmechanische Herleitung der Dielektrizitätskonstante jetzt sehr schwierig wird, doch zeigt der Verfasser, dass die Quantenkorrektionen hier unter gewissen Bedingungen unbedeutend werden, so dass man eine halbklassische Herleitung anwenden kann. Ein wichtiges Resultat ist, dass die Grenzeffekte den Energieverlust erhöhen, derselbe nimmt auf die Einheitsdicke bezogen mit abnehmender Foliendicke logarithmisch zu. Die Resultate werden auch mit experimentellen Ergebnissen verglichen, wobei jedoch eine grosse Schwierigkeit die ist, dass Metallfolien mikrokristallin aufgebaut sind.
T. Neugebauer (Budapest).

Baños, Alfredo, Jr. Normal modes characterizing magnetoelastic plane waves. Phys. Rev. (2) **104** (1956), 300–305.

In two earlier papers [Phys. Rev. (2) **97** (1955), 1435–1443; Proc. Roy. Soc. London. Ser. A. **233** (1955), 350–366; MR **16**, 1173; **17**, 921] the author has made a complete analysis of the different modes of propagation of plane waves in a uniform gaseous medium characterized by a certain velocity of sound, C. In the present paper the analysis is extended to the case of a uniform elastic solid. The modification this introduces is that the linearized form of the equation of motion

$$(1)\qquad \rho\frac{\partial^2\mathbf{v}}{\partial t^2}-\rho c^2\,\mathrm{grad}\,(\mathrm{div}\;\mathbf{v})=q\frac{\partial\mathbf{j}}{\partial t}\times\mathbf{B}$$

appropriate for the propagation of waves in a gas in the presence of a uniform magnetic field, B, is replaced by

$$(2)\quad \rho\frac{\partial^2\mathbf{v}}{\partial t^2}-\rho(V_p{}^2-V_s{}^2)\mathrm{grad}(\mathrm{div}\;\mathbf{v})-\rho V_s{}^2\nabla^2\mathbf{v}=q\frac{\partial\mathbf{j}}{\partial t}\times\mathbf{B},$$

where $\mathbf{j}$ is the current, q is the magnetic permeability and

the remaining symbols have their usual meanings. In equation (2)

$$V_p=[(\lambda+2\mu)/\rho]^{\frac{1}{2}} \text{ and } V_s=(\mu/\rho)^{\frac{1}{2}}, \tag{3}$$

are the phase velocities of compressional and shear waves for an elastic solid characterized by its Lamé moduli μ and λ. The other equations governing the problem are unaltered. By combining (2) with Maxwell's equations, the author shows that there are five distinct modes of wave propagation: The first two modes are pure shear waves one of which is a slightly attenuated shear mode with a phase velocity modified anisotropically by magneto-elastic coupling while the other shear mode is highly attenuated and exhibits a propagation constant which is essentially that of an electromagnetic wave of the same frequency. The remaining three modes are shear-compression waves of which one mode exhibits a phase velocity intermediate between the phase velocity of shear and compression waves, the second mode has a phase velocity exceeding that of the compression waves and the third shear-compression mode is again a highly attenuated wave and has propagations characteristic of an electromagnetic wave. *S. Chandresakhar.*

Pancharatnam, S. Generalized theory of interference, and its applications. I. Coherent pencils. Proc. Indian Acad. Sci. Sect. A. **44** (1956), 247–262.

Stoke's representation of an elliptically polarized beam is in terms of the intensity I and the parameters $Q=I\cos 2\beta\cos 2\chi$, $U=I\cos 2\beta\sin 2\chi$ and $V=I\sin 2\chi$, where χ is the inclination of the plane of polarization to a fixed direction in the plane containing the electric and the magnetic vectors and $\tan\beta(|\beta|\leq\pi/4)$ is numerically equal to the ratio of the axes of the ellipse traced by the end point of the electric vector and is positive or negative according as the polarization is right-handed or left-handed. It is clear that the parameters Q, U and V are, apart from the factor I, the direction cosines of a point on a sphere of unit radius — the Poincaré sphere — whose latitude and longitude are 2β and 2χ, respectively. Two opposite states of polarization (represented by similar ellipses but with their major axes at right angles to each other and with opposite sense of rotation) are represented by diametrically opposite points on the Poincaré sphere. In this paper the various theorems on the interference of polarized streams of light which were proved by Stokes [Trans. Cambridge Philos. Soc. **9** (1852), 399–416=Mathematical and physical papers, vol. 3, Cambridge, 1901, pp. 233–258] in terms of his representation are reconsidered here in terms of the Poincaré representation. Examples of such theorems are:

(1) An elliptically polarized beam represented by a point C on the Poincaré sphere when resolved into two oppositely polarized beams represented by two antipodal points A and A' on the sphere have the intensities $I\cos^2(\frac{1}{2}CA)$ and $I\cos^2(\frac{1}{2}CA')$, where CA and CA' are the great circle arcs joining C with A and A' [cf. U. Fano, J. Opt. Soc. Amer. **39** (1949), 859–863].

(2) If two coherent beams of elliptically polarized light of intensities I_1 and I_2 and represented by points A and B on the Poincaré sphere are combined, then the resultant beam has the intensity

$$I=I_1+I_2+2(I_1I_2)^{\frac{1}{2}}\cos(\tfrac{1}{2}AB)\cos\delta,$$

where δ is the assigned constant difference in the phases of the two beams; further, the resultant beam is represented by a point C on the Poincaré sphere such that

$$\frac{I_1}{I}=\frac{\sin^2(\frac{1}{2}BC)}{\sin^2(\frac{1}{2}AB)},\quad \frac{I_2}{I}=\frac{\sin^2(\frac{1}{2}AC)}{\sin^2(\frac{1}{2}AB)}.$$

S. Chandrasekhar (Williams Bay, Wis.).

★ **Альперт, Я. Л.; Гинзбург, В. Л.; и Фейнберг, Е. Л. [Al'pert, Ya. L.; Ginzburg, V. L.; i Feĭnberg, E. L.] Распространение радиоволн. [The propagation of radio waves.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 883 pp. 33.65 rubles.

This book, which consists of 870 pages of textual material and an additional 12 pages of bibliographic material, is possibly the most complete survey of radiowave propagation up to the time of publication in 1953. The three authors are well-known Russian physicists who have worked in propagation theory for many years. The book is divided into four parts. The first part, written by E. L. Feinberg and concerned with propagation along the earth's surface comprises 10 chapters and 277 pages. The material covered in this part starts with the basic equation of electromagnetic theory and continues to treat more and more difficult problems until chapter 10 covers propagation over the earth's sphere. The second part, written by V. L. Ginzburg, and concerned with ionospheric propagation comprises 7 chapters and 270 pages. This section is a theoretical treatment of ionospheric propagation. Parts 3 and 4 were written by Ya. L. Al'pert and are concerned with the experimental investigations of radiowave propagation. Each part is self-contained and can be used independently of the others. Since each author apparently wrote his section with this idea in mind, there is a great deal of repetition and a lack of coherence between the sections. However, the book as a whole is a veritable treasure house of material on propagation. Amazingly enough, while most of the theoretical advances are credited to Soviet Scientists, the major share of the curves, figures and other experimental results are primarily of Western origin. The authors seem to have culled the literature extensively and to have selected only the best material for inclusion in this book. *M. D. Friedman* (Newtonville, Mass.).

Gosar, P. Multiple small angle scattering of waves by an inhomogeneous medium. Nuovo Cimento (10) **4** (1956), 688–702.

The scattering of a light wave in a medium with randomly distributed small density fluctuations is studied. To a first approximation (the "Rayleigh-Gans-Born approximation") the scattering cross-section is determined by the autocorrelation of the fluctuations in the dielectric constant, $\langle\Delta\varepsilon(\mathbf{r})\Delta\varepsilon(\mathbf{r}')\rangle$ [see P. Debye and A. M. Bueche, J. Appl. Phys. **20** (1949), 518–525]. The present paper improves upon this by taking into account the variation of phase of the primary wave, due to the local density variations. [For analogous suggestions for improving upon the Born approximation in quantum mechanics see, e.g., H. J. Groenewold, Mat.-Fys. Medd. Danske Vid. Selsk. **30** (1956), no. 19; MR **18**, 851]. The autocorrelation of $\Delta\varepsilon(\mathbf{r})$ is then to be replaced with the autocorrelation function of $(\Delta\varepsilon(\mathbf{r})\exp[i\varphi(\mathbf{r})])$, where $\varphi(\mathbf{r})$ is the local phase of the primary wave. Moreover, by using the small angle approximation, the attenuation of the primary beam can be found from an integro-differential equation of the type of a transport equation.

N. G. van Kampen (Utrecht).

Popovkin, V. I. Application of a variational method to solution of a problem on diffraction of two-dimensional cylindrical electromagnetic waves at an opening in a conducting screen. Kazan. Aviac. Inst. Trudy **29** (1955), 47–68. (Russian)

This paper deals with the problem of radiation from a rectangular waveguide through a rectangular opening in an infinite screen perpendicular to the axis of the waveguide. The method used is a variational one, developed by G. V. Kisunko [Dokl. Akad. Nauk SSSR (N.S.) **66** (1949), 863–866; MR **10**, 764]. The standing wave ratio in the waveguide is calculated as a function of frequency for aperture half as big as the waveguide cross-section.

The method used does not compare favorably with the Schwinger variational method. Kisunko's method involves the actual approximate calculation of the aperture field, from which the reflection coefficient is then calculated; it does not give (as Schwinger's method does) an expression for the reflection coefficient as a functional of the aperture field, stationary with respect to variations in the aperture field. *J. Shmoys* (Brooklyn, N.Y.).

Anisimov, E. V.; and Sovetov, N. M. Propagation of electromagnetic waves along a spiral ribbon inside a circular wave-guide. Ž. Tehn. Fiz. **25** (1955), 1965–1971. (Russian)

The authors deal with the problem of propagation of electromagnetic waves along a perfectly conducting sprial ribbon of zero thickness inside a perfectly conducting cylinder. The ribbon is treated as an obstacle in a cylindrical surface. By explicitly recognizing the symmetry of the field resulting from the geometry of the boundary, the authors reduce the problem to a two dimensional one, extending over one period of the helix. The problem is solved by assuming the ribbon to be very narrow (compared to wave length). [Similar treatment and results can be found in L. Stark, Mass. Inst. Tech., Lincoln Lab. Tech. Rep. no. **26** (1953).] *J. Shmoys.*

Lindroth, Kristen. Reflection of electromagnetic waves from thin metal strips (passive antennae). Kungl. Tekn. Högsk. Handl. Stockholm no. **91** (1955), 62 pp.

The paper deals with the scattering of a plane electromagnetic wave by a perfectly conducting narrow strip. The approach used is that Hallén [Cruft Lab., Harvard Univ., Tech. Rep. no. **49** (1948)], and the iteration procedure has been carried out to three terms. *J. Shmoys.*

Vaĭnšteĭn, L. A. Electromagnetic surface waves over a comb-shaped structure. Ž. Tehn. Fiz. **26** (1956), 385–397. (Russian)

The paper deals with the propagation of a surface wave on a structure consisting of parallel, equidistant, identical, perfectly conducting infinite strips built up on an infinite perfectly conducting plane perpendicular to the strips. The propagation problem is reduced to an infinite set of equations in an infinite number of unknowns; for most purposes, however, it is sufficient to consider only two unknowns. Numerical results are presented and discussed in detail. The method used and some of the result are closely related to those given by R. A. Hurd [Canad. J. Phys. **32** (1954), 727–734; MR **16**, 773]. *J. Shmoys.*

★**Stern, Thomas Edwin. Piecewise-linear network theory.** Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Mass., Tech. Rep. 315 (1956), ii+76 pp.

A systematic treatment of resistive electric networks with piecewise-linear characteristics is presented; it provides a link between general linear network theory as exemplified by Zadeh's work [J. Franklin Inst. **255** (1953), 387–408; MR **15**, 376] and conventional methods of circuit design. An algebra of inequalities is developed permitting compact, in general not unique, symbolic representation of piecewise-linear functions; combined with corresponding symbolic description of circuit elements, it provides a ready tool for the analysis of general topological circuits containing resistances and idealized diodes and tubes. Basic to the synthesis procedures is decomposition in terms of elementary monotonic convex (concave) characteristics. Driving point synthesis leads to Foster- and Cauer-type analogs, while transfer synthesis is predicated on proper interpolation procedures by means of tesselation with respect to $(n-1)$-regions in Euclidean n-space and on the synthesis of requisite unit function generators. Examples include the approximative electrical generation of a thermodynamic surface. *H. G. Baerwald* (Cleveland, Ohio).

See also: Korobkov, p. 115; Dingle, p. 133; Brittin, p. 190; Longmire and Rosenbluth, p. 191; Atoji, p. 199; Wheelon, p. 204; Malyužinec, p. 207; Talwar and Abbi, p. 207; Sykes, p. 207; Sokolov, p. 217; Štraskevič, p. 225.

Classical Thermodynamics, Heat Transfer

Landsberg, P. T. Foundations of thermodynamics. Rev. Mod. Phys. **28** (1956), 363–392.

"The purpose of this paper is to effect a geometrization of thermodynamics by a development of the axiomatic approach initiated by Carathéodory. Most of the material presented appears here at least in a new form. It is as a result of this geometrization of thermodynamics that the third law appears in a novel form.

"The paper deals throughout with sets of points in a suitably defined n-dimensional thermodynamic phase space E. Thus, for properly restricted ranges of the independent variables of a given system, there will exist a set of points β with the following properties: for any two points A, B of β, there exists an adiabatic linkage between the states which these points represent. The first law of thermodynamics is concerned largely with the properties of such sets of points β. For any given set of points β, one is now led to look for those subsets of β (if they exist) which fill an n-dimensional volume in E. The second law deals only with such subsets, which are denoted by γ, and it is essential that these subsets γ be open (i.e., they must exclude boundary points). When one has finished with the second law, certain questions are left unanswered. For instance, can a set β contain the boundary points of any of its subsets γ? Can a set β contain points which are neither in any of its subsets γ nor on the boundaries of such subsets? A third law of thermodynamics becomes, therefore, absolutely essential in the present approach if ambiguities are to be avoided. It settles the two questions which have just been raised.

"This paper aims not only at developing these new concepts, but also at removing certain deficiencies of the axiomatic approach as currently expounded." These include: (1) Ambiguity in definitions, such as quasistatic, reversible, diathermanous, and adiabatic. (2) Mathematical difficulties arising from the use of an unbounded space. (3) Conventional treatments do not deal with any of the formulations of the third law.

An outstanding feature of the present development is that the common issues, and there possible resolutions, regarding absolute zero temperature and a Third Law are formulated in a sharp and decisive manner.

C. C. Torrance (Monterey, Calif.).

Vodička, Vaclav. Stationary temperature fields in a two-layer plate. Arch. Mech. Stos. 9 (1957), 19–24. (Polish and Russian summaries)

Formulas are derived for steady-state temperatures $u_1(x, y)$ and $u_2(x, y)$ in the two layers $y_1<y<y_2$ and $y_2<y<y_3$, respectively, of a composite slab of material that fills the region $-\infty<x<\infty$, $y_1<y<y_3$. The temperatures prescribed at the external faces $y=y_1$ and $y=y_3$ are arbitrary periodic functions of x, with period 2π. Each layer of material is assumed homogeneous with its individual thermal coefficients, and thermal resistance is assumed at the interface $y=y_2$. The author's formulas represent the temperature functions $u_1(x, y)$ and $u_2(x, y)$ by infinite series found by the classical method of separation of variables.

R. V. Churchill.

Selig, F.; und Fieber, H. Wärmeleitprobleme mit zeitlich variabler Übergangszahl. Österreich. Ing-Arch. 11 (1957), 37–40.

Consider the heat equation for a region in a plane and a boundary condition of the type $dT/dn+h(t)T=0$ on a fixed curve Γ, where T denotes temperatures, dT/dn a normal derivative, and $h(t)$ denotes a prescribed function of time. The authors present a transformation of coordinates, involving the time coordinate t as well as space coordinates, that transforms the heat equation into one similar to the original equation and replaces the coefficient $h(t)$ by a constant α; but the boundary Γ is replaced by a time-dependent curve. Thus the transformation substitutes a moving boundary for a time-dependent coefficient of heat transfer at the boundary. Examples and generalizations are mentioned.

R. V. Churchill.

Goody, R. M. The influence of radiative transfer on cellular convection. J. Fluid Mech. 1 (1956), 424–435.

The onset of thermal instability in a plane parallel atmosphere in radiative equilibrium and heated from below is considered. The essential modification from the case generally considered [cf. Rayleigh, Phil. Mag. (6) **32** (1916), 529–546; also Pellew and Southwell, Proc. Roy. Soc. London. Ser. A. **176** (1940), 312–343; MR **2**, 266] consists in that one must take into account the effects of radiative heating. Explicit formulae for the latter effects are obtained by solving the equation of radiative transfer (for a gray atmosphere) appropriately in two limiting cases. The resulting characteristic value problem for the Rayleigh number is solved by a variational method. The author concludes that "the effect of radiative transfer both on the initial static state and on the dynamical equations is such that the fluid is stabilized."

S. Chandrasekhar (Williams Bay, Wis.).

Signorini, Antonio. Sulla propagazione stazionaria del calore attraverso a un involucro separante due ambienti a temperatura uniforme. Abh. Math. Sem. Univ. Hamburg **21** (1957), 78–86.

The object of this paper is to furnish upper and lower bounds to the quantity of heat which, in the state of steady flow, crosses in unit time a shell which separates two media at different constant temperatures.

E. T. Copson (St. Andrews).

White, G. W. T. On the use of matrices for solving periodic heat flow problems. Appl. Sci. Res. A. 6 (1957), 433–444.

Consider those solutions of the heat equation

$$D^{-1}\nabla^2 V=D^{-1}\partial V/\partial\tau \tag{1}$$

that can be written in the form $V=U(p)\exp(i\omega\tau)$, where the function U is independent of τ and satisfies the ordinary differential equation

$$d^2U/dp^2+p^{-1}(n-1)\,dU/dp+\delta^2U=0.$$

Here $p=x$, $p^2=x^2+y^2$, or $p^2=x^2+y^2+z^2$ according as $n=1$, 2, or 3; and the constant $\delta^2=-i\omega/D$. The author calls $W(p)=-KdU/dp$, where K is the thermal conductivity, the heat flow. In Part I of the paper it is proved that at two different points of a homogeneous medium the functions U, W are related by the equation (2) below, in which the determinant of the matrix A has the value $(p_2/p_1)^{n-1}$. [For $n=1$, 2 similar results were obtained by A. H. van Gorcum, Appl. Sci. Res. A. 2 (1950), 272–280; MR **12**, 709. Also see papers by V. Vodicka, ibid. 5 (1955), 108–114, 115–120; MR **16**, 710.] Part II of the paper is devoted to deriving the matrix representation (2) of temperature waves. Here temperature waves are those solutions of equation (1) which can be written in the forms $V_+=U_+(p)\exp(i\omega\tau-ikp+e)$, a forward traveling wave, and

$$V_-=U_-(p)\exp(i\omega\tau+ikp+e),$$

a reverse wave. The letter e stands for a phase constant, and U_+, U_- are real functions of p. Such representations of V exist for $n=1$ and 3, but not for $n=2$. The elements of the matrix B are derived in two cases: one, when p_1 and p_2 are within the same homogeneous medium, and, two, when p_1 and p_2 are close to each other but on opposite sides of an interface separating two different homogeneous media. The author states that one can find the relation between the values of the forward and reverse waves in different parts of a solid consisting of many adjacent layers through the simple expedient of matrix multiplication,

$$\begin{pmatrix} U(p_1)\\ W(p_1)\end{pmatrix}=A\begin{pmatrix} U(p_2)\\ W(p_2)\end{pmatrix}\cdot\begin{pmatrix} V_+(p_1,\tau)\\ V_-(p_1,\tau)\end{pmatrix}=B\begin{pmatrix} V_+(p_2,\tau)\\ V_-(p_1,\tau)\end{pmatrix}. \tag{2}$$

F. G. Dressel (Durham, N.C.).

See also: Wooding, p. 208.

Quantum Mechanics

★ **Destouches, Jean-Louis. La quantification en théorie fonctionnelle des corpuscules.** Gauthier-Villars, Paris, 1956. vi+141 pp. 2,000 fr.

In the first chapter it is argued that a particle should be represented by a space function rather than by a single point, in order to allow for the interaction with the surrounding universe. This function u is supposed to have an objective existence and may be visualized in terms of a fluid. The hydrodynamical equations for this fluid lead to an equation of motion for u, which reduces in linear approximation to the Schrödinger equation. In the second chapter a restriction is imposed on the nonlinear terms of the equation by postulating that solutions of a certain type exist, corresponding to plane waves in quantum mechanics.

In chapters 3 and 4 the functions u are "quantized" by postulating an additional "quantization condition" S (in analogy with Planck's phase integral condition). However, functions u not satisfying S are not rejected; they are supposed to describe transient effects, which are important during quantum jumps. For correspondence with the usual quantum mechanics it will be desirable to choose S in such a way, that the energy values connected with the "quantized" solutions u are the usual ones. Accordingly, chapters 5 and 6 deal with an analytical device for characterizing spectra. (It consists mainly in defining an integral function whose zeros are the eigenvalues of the spectrum). In the last chapter these general ideas are applied to a few simple examples and the effect of the nonlinear terms is discussed.

The aim of this "functional theory of corpuscles" is, firstly, to find a deterministic foundation of quantum mechanics, in the same vein as de Broglie's "theory of the double solution" [Une tentative d'interprétation causale et non linéaire de la mécanique ondulatoire, Gauthier-Villars, Paris, 1956]. Secondly, it is hoped that the modifications resulting from the nonlinear terms not only provide the features required for a deterministic interpretation of u, but will also be helpful in nuclear theory. So far the theory is rather sketchy, as neither the nonlinear terms, nor the form of the quantizing condition, can be specified. *N. G. van Kampen* (Utrecht).

Gürsey, Feza. New algebraic identities and divergence equations for the Dirac electron. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **21** (1956), 85–95. (Turkish summary)

The quaternion formalism for describing a Dirac spinor field developed in previous papers [same Rev. **20** (1955), 149–171; **21** (1956), 33–54; MR **17**, 1014] is used to derive algebraic and differential identities satisfied by tensors formed from the Dirac wave functions. *A. H. Taub.*

Hill, E. L. Function spaces in quantum-mechanical theory. Phys. Rev. (2) **104** (1956), 1173–1178.

Von Neumann's attempt to provide a clear mathematical basis for quantum mechanics took the space L_2 of normalizable functions as the scene of operations. Current field theory deals chiefly in unnormalizable functions. The present paper, which is both suggestive and illuminating, faces this dilemma squarely. The author's summary is as follows: "The use of wave functions which are not normalizable in the sense of quadratic integrability introduces concepts which are difficult to incorporate into the statistical interpretation of quantum mechanical theory. A study is made of other inner-product function spaces which can be defined on the basis of the asymptotic behaviour of various classes of functions. Only the cases of plane waves, cylindrical waves, and spherical waves are considered in detail. The theory offers an alternative approach to the treatment of open systems which is closer to classical field theories than is that of von Neumann." *A. J. Coleman* (Toronto, Ont.).

Holmberg, Bengt. On the separation of variables for a three-body quantummechanical system. Kungl. Fysiogr. Sällsk. i Lund Forh. **26** (1956), no. 14, 10 pp.

Starting with the Schroedinger equation for three particles in a coordinate system in which their center of mass is at rest, the author carries out a separation of variables, obtaining a system of differential equations in which the independent variables are the distances between the particles. He shows that by a change of axes these equations can be made to go over into those obtained by C. F. Curtiss, J. O. Hirschfelder and F. T. Adler [J. Chem. Phys. **18** (1950), 1638–1642; MR **12**, 782]. *N. Rosen* (Haifa).

Sokolov, A.; and Kerimov, B. On the scattering of particles by a force centre according to the radiation damping theory. Nuovo Cimento (10) **5** (1957), 921–939.

The method of radiation damping theory is applied to the problems of the scattering of a Dirac particle by a delta function potential and the scattering of a Klein-Gordon particle by a short-range, Yukawa and square well potential. Explicit results are obtained and compared, in the case of a delta function potential, with the exact solution. *H. Feshbach* (Cambridge, Mass.).

Hecht, Charles E.; and Mayer, Joseph E. Extension of the WKB equation. Phys. Rev. (2) **106** (1957), 1156–1160.

This paper derives a scheme which extends the so-called Wentzel-Kramers-Brillouin solution of the Sturm-Liouville problem in two ways: first, they have a scheme for mapping the whole solution including turning-points and exponential regions on the solutions $\sin kx$ of the bounded, constant potential, without turning-point singularities, and second, they give an iteration procedure for finding the mapping function to arbitrarily high accuracy. *P. W. Anderson* (Murray Hill, N.J.).

McLennan, J. A., Jr. Conformally invariant wave equations for non-linear and interacting fields. Nuovo Cimento (10) **5** (1957), 640–647.

By constrcting Lagrangean functions which are invariant under inversions and dilations as well as translations and Lorentz transformations the author determines non-linear conformally invariant wave equations for spinors of arbitrary rank. In addition, conformally invariant equations for interacting fields are given. *A. H. Taub* (Urbana, Ill.).

Raman, Varadarata Venkata. Discussion du phénomène du tremblement de Schrödinger en termes de l'hydrodynamique relativiste de M. Takabayasi. C. R. Acad. Sci. Paris **244** (1957), 1155–1157.

The author computes various tensor quantities from plane wave solutions of the Dirac equation for an electron. The wave functions used are mixtures of positive and negative energy states. *A. H. Taub.*

Gürsey, F. General relativistic interpretation of some spinor wave equations. Nuovo Cimento (10) **5** (1957), 154–171.

The author discusses the torsion tensor for a special tetrad of orthogonal vectors in a conformally flat space-time and derives two vectors from this tensor. The requirement that both of these vectors vanish or one of them does so is then studied in terms of the equations satisfied by the two-component two index spinors corresponding to these vectors. The analogy between the differential equation obtained in the first case and the Dirac equation for a particle of spin $\frac{1}{2}$ and zero mass is remarked upon. In the second case the spinor differential equation is shown to be similar to a non-linear equation proposed by Heisenberg. *A. H. Taub* (Urbana, Ill.).

Riesenfeld, W. B.; and Watson, K. M. Energy of a many-particle system. Phys. Rev. (2) **104** (1956), 492–510.

The problem of the energy eigenvalue of a many-particle system is considered from the point of view of applications to statistical mechanics. The general perturbation method used for this purpose has been described in a previous paper on the application of scattering theory to statistical mechanics [K. M. Watson, Phys. Rev. (2) **103** (1956), 489–498; MR **18**, 82]. This perturbation method leads to an integral equation of the form $M=1+a^{-1}(V-O)M$, where $a=E+K-O$, E is a complex variable corresponding to the energy, K is the kinetic energy, and O is an arbitrary operator introduced for the purpose of constructing the integral equation. According to the choice of this operator, different perturbation methods are obtained (Wigner-Brillouin, Francis-Watson, etc.). However, these methods do not permit simple perturbation expansions for the many particle system. To overcome this difficulty an "optical model" method is formulated where the integral equation is written in the form $M=1+a^{-1}PVM$, P being an operator with suitable properties. The conditions are investigated under which this is possible. The equations used by Brueckner and his collaborators (Phys. Rev. (2) **103** (1956), 1008–1016; MR **19**, 99) for the treatment of nuclear many-body problems are obtained by choosing $P=(1-\Lambda_{p_0})P_{ND}$ and $O=\Delta_0$; here Λ_{p_0} is a projection operator on the initial unperturbed state $|p_0\rangle$, and P_{ND} is an operator which reduces the diagonal elements of a matrix to zero, but does not affect the non-diagonal elements. Δ_0 is defined by the equation $(1-P)VM|p_0\rangle=\Delta_0 M|p_0\rangle$. The diagonal elements of this operator represent the energy shift of the p_0-state. The level shift for the many-particle system is then given by the expression $\langle p_0|VM|p_0\rangle$ which is expanded in terms of two-body scattering operators. The terms in this expansion may be classified according to their "linked clusters" as defined by Brueckner, but the authors propose instead a "nearest neighbor" expansion where the lowest term describes an interaction of pairs of particles, the next triples of particles, etc. The rapidity of convergence of this expansion depends primarily on the particle density, and on the ratio of the strength of the interaction to an "effective energy of excitation", which may be of the order of the thermal or zero point energy. The method is applied to the calculation of the equation of state at low density and temperature. *E. Gora.*

Bopp, Fritz. La mécanique quantique est-elle une mécanique statistique classique particulière? Ann. Inst. H. Poincaré **15** (1956), 81–112.

The possibility is discussed of replacing quantum mechanics by a statistical theory where probability amplitudes would no longer play any role. A way to achieve this aim is presented in the simplest case of a particle obeying the Schrödinger equation in one dimension. One replaces the statistical matrix of von Neumann by a distribution function $f(p, q, t)$ in phase space, defined as the expectation value of the statistical matrix for a gaussian wave packet centered around p and q. The spatial width l of this packet is taken the same for all p, q and plays the role of a fundamental length in the theory. The function f is shown to obey for general potential an integro-differential equation determining its time evolution from its value at an initial time. This equation is called statistical equation of motion and takes the place of the Schrödinger equation as basic equation of the theory. Of course, not every distribution f corresponds to a possible density matrix, but if this property holds at one time the statistical equation insures its validity at all times. The statistical theory is meant to allow only such functions f which correspond to a statistical matrix and efforts are made to find for this restriction an intrinsic formulation. Mean values and mean square fluctuations of observable quantities are considered and a short discussion is given of diaphragm experiments.
L. Van Hove (Utrecht).

Aymard, Alix. Les équations de l'électron magnétique déduites de la théorie des champs de tétrapodes. C. R. Acad. Sci. Paris **243** (1956), 1100–1102.

The concepts of a previous note [same C. R. **243** (1956), 885–888; MR **18**, 362] are used to construct a model of the spinning electron. *L. Van Hove.*

Aymard, Alix. La quantification du moment cinétique déduite de la théorie des champs de tétrapodes. C. R. Acad. Sci. Paris **244** (1957), 312–313.

A field-theoretical formalism developed earlier and previously suggested as a model of spinning electron [same C.R. **243** (1956), 885–888, 1198–1201; MR **18**, 542; see also the paper reviewed above] is here shown to lead to quantization of the angular momentum.
L. Van Hove (Utrecht).

Green, H. S. Separability of a covariant wave equation. Nuovo Cimento (10) **5** (1957), 866–871.

It is shown that Wick-Cutkosky [Wick, Phys. Rev. (2) **96** (1954), 1124–1134; MR **16**, 655; Cutkosky, ibid. **96** (1954), 1135–1141; MR **16**, 656] covariant equation for binding of two bosons is a separable equation if bipolar coordinates are used. More explicitly, write

$$p_1=p_s\sin\phi\cos\theta,\ p_2=p_s\sin\theta\sin\phi,\ p_3=p_s\cos\theta,$$
$$p_4=c\sin\alpha/(\cos\alpha-\cos\beta),$$
$$p_s=c\sin\beta/(\cos\alpha-\cos\beta).$$

Here p_1, p_2, p_3, p_4 is the relative energy-momentum vector, $c^2=m_1^2-E_1^2=m_2^2-E_2^2$. The wave-function $\Psi(p)$ can then be written as $\Psi=f(\alpha)g(\beta)Y_{e,m}(\theta,\phi)/p_s$. The separability of the equation implies that a new operator $(-\partial^2/\partial\beta^2+l(l+1)/\sin^2\beta)$ exists which commutes with total energy. The author conjectures that the eigenvalues of this operator may have connection with the "strangeness" quantum-number. To the reviewer, it is not clear whether the separability of the Wick-Cutkosky equation is a consequence of the "ladder approximation" made by these authors or is a general property of such co-variant equations. *A. Salam* (London).

Jauch, J. M. Covariant hyperquantization. Helv. Phys. Acta **29** (1956), 287–312.

The formalism of "hyperquantization" of vacuum expectation values of chronological products of field operators recently studied by several authors [see F. Coester, Phys. Rev. (2) **95** (1954), 1318–1323, and references quoted in this paper; MR **16**, 320] is developed here from a rigorously mathematical point of view. The chief merit of the paper lies in proposing an operator formalism which is fully covariant under the Lorentz group. This was not true of the earlier formulations [Coester, loc. cit.]
A. Salam (London).

Livšic, M. S. On the scattering matrix of an intermediate system. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 67–70. (Russian)

The author considers the scattering process involved in the elastic collision of two uncharged particles with zero angular momentum and zero spins, allowing for the relativistic dependence of mass on velocity. The process is thought of as $a_1+a_2\to C\to a_2+a_1$, where C is called the 'intermediate' or 'composite' system.

The author uses separate wave functions in the internal region $r<R$ and the external region $r>R$:

$$u(r)=\frac{a_0}{r}\{e^{-ipr/\hbar}-S(W)e^{ipr/\hbar}\}\quad (r>R),$$

$$v(r)=\sum_{k=1}^{\infty} f_k v_k(r)\quad (r<R).$$

The Hamiltonian is taken in the form

$$\begin{bmatrix} W & \Gamma \\ \Gamma^* & A \end{bmatrix},$$

where $W=c\{\sqrt{(m_1^2c^2+p^2)}+\sqrt{(m_2^2c^2+p^2)}\}$ is the energy operator (relative to the mass centre) for the external region, A is the energy operator for the internal region, having $\{v_k(r)\}$ as a complete orthonormal system of wave functions, and Γ can be interpreted as determining the probabilities of the splitting up and the formation of the intermediate system.

The author shows that the scattering 'matrix' $S(W)$ (here a scalar) can be expressed in the form

$$S(W)=e^{-2ipr/\hbar}\, S_0(W),$$

where $S_0(W)$ is the characteristic matrix function [M. S. Livšic, Mat. Sb. N.S. **34(76)** (1954), 145–199; MR **16**, 48] of a certain non-self-adjoint operator H. He deduces that

$$S_0(W)=\frac{1+\frac{1}{2}iP\hbar\varphi(W)}{1-\frac{1}{2}iP\hbar\varphi(W)},$$

where P is the total probability that the interacting system breaks up in unit time, and

$$\varphi(W)=\int_a^b \frac{d\sigma(t)}{t-W},$$

where $\sigma(t)$ is a non-decreasing function of the form $(E_t l_0, l_0)$, E_t being the spectral resolution of the self-adjoint operator A, and l_0 a certain unit vector.

Finally, the author indicates how these results make it possible to reconstruct a matrix representation of the energy operator A of the intermediate system from the scattering matrix $S(W)$. This process is carried out for the Heisenberg model, and it turns out that A has a continuous spectrum filling the interval $(-2mc^2, 2mc^2)$.

Only outline proofs are given. *F. Smithies.*

Livšic, M. S. The method of non-selfadjoint operators in scattering theory. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 212–218. (Russian)

The author considers the scattering of an incident particle by an atomic nucleus, using the formalism in which the incident particle and the target nucleus are thought of as forming a compound nucleus C, which later decays into a residual nucleus and an emitted particle. The paper forms a continuation of the work reported in the paper reviewed above. It is now supposed that there are m possible modes or 'channels' Ω_α $(1\leq\alpha\leq m)$ of formation or decay of C, and that in the channel Ω_α the residual nucleus X_α and the emitted particle a_α do not interact when $r_\alpha>R_\alpha$; the effects of spin are neglected.

Taking for the external wave function for the initial channel Ω_α an m-component vector $u^{(\alpha)}$ given by equations of the form

$$u_\beta^{(\alpha)}(r_\beta)=-c_\alpha\frac{S_{\alpha\beta}(W)e^{iK_\beta r_\beta}}{\sqrt{(4\pi V_\beta)}}\varphi_\beta\chi_\beta\quad(\beta\neq\alpha),$$

$$u_\alpha^{(\alpha)}(r_\alpha)=c_\alpha\frac{e^{-iK_\alpha r_\alpha}-S_{\alpha\alpha}(W)e^{iK_\alpha r_\alpha}}{\sqrt{(4\pi V_\alpha)}}\varphi_\alpha\chi_\alpha,$$

where $S(W)=[S_{\alpha\beta}(W)]$ is the scattering matrix, and φ_β, χ_β are eigenfunctions of the particle a_β and the nucleus X_β respectively, the author shows that the wave equation for the whole system leads to an equation of the form

$$(1)\qquad H^*f^{(\alpha)}-Wf^{(\alpha)}=-\tfrac{1}{2}i\sqrt{\left(\frac{h}{\pi}\right)}d_\alpha e^{(\alpha)},$$

where H is a non-Hermitian linear operator in an appropriate Hilbert space of which $f^{(\alpha)}$ and $e^{(\alpha)}$ are elements. Equation (1) may be regarded as a wave equation having complex energy levels. The non-Hermitian rank of H is exactly m, the number of possible decay channels; the vectors $e^{(\alpha)}$ span the non-Hermitian subspace of H, and $\|e^{(\alpha)}\|^2=\omega_\alpha$ is the probability per unit time that C will decay along the channel Ω_α.

If the scattering matrix is expressed in the form

$$S(W)=[e^{-iK_\alpha R_\alpha}\delta_{\alpha\beta}]S_C(W)[e^{-iK_\alpha R_\alpha}\delta_{\alpha\beta}],$$

it turns out that

$$S_C(W)=I+i[((H^*-WI)^{-1}e^{(\alpha)}, e^{(\beta)})],$$

which is the characteristic matrix function of the operator H, in the sense of the author's general theory of non-Hermitian operators [Mat. Sb. N.S. **34(76)** (1954), 145–199; MR **16**, 48]. The following results now follow from the theory in the case when the $e^{(\alpha)}$ are independent of W: (i) $S_C(W)$ determines H up to unitary equivalence, and therefore provides a complete description of the behaviour of the compound nucleus; (ii) $S_C(W)$ is an analytic function of W in the upper half-plane; (iii) the Hermitian matrix $I-S_C(W)S_C^*(W)$ is non-negative definite; (iv) the eigenvalues $W_k=E_k+\frac{1}{2}i\Gamma_k$ of H lie in the upper half-plane and are precisely the zeros of det $S_C(W)$; (v) the continuous spectrum (if any) of H lies on the real axis, and an interval (a, b) is free of points of the spectrum if and only if $S_C(W)$ can be analytically continued across (a, b); for $a<W<b$, the matrix $S_C(W)$ is unitary.

Similar but slightly more complex results are obtained in the case when the $e^{(\alpha)}$ are not independent of W.

F. Smithies (Cambridge, England).

Good, R. H., Jr. Particle aspect of the electromagnetic field equations. Phys. Rev. (2) **105** (1957), 1914–1919.

The dual interpretation of quantum mechanical theory in terms of waves and particles is reflected in a dual method of formulating the partial differential equations of linear field theories which are invariant under the Lorentz group. The wave aspect is expressed most simply by the study of particular solutions showing wave-like properties. The particle aspect requires more subtle interpretation in terms of the transformation properties of the field under Lorentz transformations. While this general theory is well known to both mathematicians and theoretical physicists, the variety of ways in which it can be formulated has led to many overlapping discussions in the literature.

The paper reviewed here deals with the Maxwell-Lorentz field equations in free space. The field is expressed

in the manner used by L. Silberstein and by H. Bateman [The mathematical analysis of electrical and optical wave motion⋯, Cambridge, 1915], writing $\psi=E+iB$. The three components of this vector are treated as a 3×1 matrix. The analysis is restricted primarily to radiation fields which are expressible in terms of plane waves of non-vanishing frequency, which the author considers to form a "complete set", following the usual fiction of quantum field theories. On this basis the transformation properties and conservation properties of the field under the Lorentz group are established. The advantage of this formulation is that it emphasizes the connections between energy and frequency, momentum and de Broglie wave-length, spin and polarization, of the field, while the densities are related to expectation values of operators.

E. L. Hill (Minneapolis, Minn.).

Sokolov, A. On the relativistic motion of electrons in magnetic fields when quantum effects are taken into account. Nuovo Cimento (10) **4** (1956), supplemento, 743–759.

This paper reviews the theory of the emission of radiation by electrons moving in magnetic fields, with particular attention to the case of motion in nearly circular orbits. The main problem of principle is to determine the realm of validity of the classical theory and to describe the nature of properly quantum mechanical effects when they occur. After a historical review, the author considers the classical problem of the motion of an electron in a cylindrically symmetrical magnetic field which he chooses of a special form so as to provide focussing. He introduces a set of adiabatic invariants for the motion and expresses the amplitude of radial and axial oscillations in terms of them. He then discusses the motion of a Dirac electron in a homogeneous magnetic field and derives expressions for intensity of radiation emitted by such a particle and for the rate of change of its radial adiabatic invariant. By modifying the formulae to include the effects of the above mentioned inhomogeneous field, expressions are obtained for the amplitude of radial and axial oscillations induced by the emission of radiation.

A. S. Wightman (Princeton, N.J.).

Kuni, F. M. On the dispersion relation for nucleon-nucleon scattering. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 571-574. (Russian)

The author starts from the expression for the scattering amplitude of a two nucleon system in terms of interaction representation operators and no-particle state. By a familiar sequence of operations [see, e.g., F. Low, Phys. Rev. (2) **97** (1955), 1392–1398], he transforms this expression into a vacuum expectation value of Heisenberg operators. He remarks that the time ordered product can be replaced by a totally retarded product. Then comes the main point of the note. If dispersion relations are written for this retarded amplitude, they contain an integral of an absorptive amplitude over an "unphysical region" in which the momenta occurring are complex. The author asserts that for singlet states of the incoming nucleons the absorptive amplitude vanishes in the unphysical region (apart from the effect of possible intermediate bound states which are not considered), so that the dispersion relation contains only measurable amplitudes. To the reviewer, it seems that the argument as it stands only applies to forward scattering, and even in that case is inconclusive because it uses for complex momenta an expression which is derived only for real physical momenta. In any case, the result is wrong because intermediate states containing only π-mesons contribute non-zero absorptive amplitudes in the non-physical region.

A. S. Wightman (Princeton, N.J.).

★ **Jauch, J. M.; and Rohrlich, F. The theory of photons and electrons. The relativistic quantum field theory of charged particles with spin one-half.** Addison-Wesley Publishing Company, Inc., Cambridge, Massachusetts, 1955. xiv+488 pp. $10.00.

This is a treatise and a textbook on the quantum electrodynamics of spin one half particles. It is a readable representation of the subject as it stood shortly after the advances associated with the names Tomonaga, Schwinger, Feynman, and Dyson. There are sixteen chapters and seven appendices. The first chapter, begins with a few preliminaries of notation and some basic facts of quantum mechanics and relativity. Then the notion of a local field is introduced and it is shown how Schwinger's invariant action principle gives rise to equations of motion and conservation laws. The consistency of the standard commutation and anti-commutation relations with the action principle is shown. The second and third chapters are devoted to the theory of the free photon field and free electron field respectively. The material is standard, the only innovation being a discussion of the quantum mechanical observables corresponding to the Stokes parameters of a photon. In chapter four the coupling between electron and photon field is introduced and the interaction picture defined. In chapter five the invariance properties of the coupled fields under Lorentz transformations, gauge transformations, and charge conjugation are discussed. Chapter six deals with the special circumstances arising from the zero mass of the photon: the Coulomb interaction and auxiliary condition. The latter is handled by several different methods. The first six chapters set a general style which is maintained throughout the book. It is characterized by covariance of notation and elegance of exposition. {However, in the reviewer's opinion, the first six chapters have a basic defect which is, at least in part, a faithful reflection of the state of the subject at the time the book was written: they pretend to be deductive, even when the subject matter admits at most heuristicism. The patient reader finds out in chapter 9, after 169 pages, that divergence difficulties exist, and in chapter ten how to deal with them by the method of renormalization, but he never does find out which arguments of the preceding general theory are swindles and which reliable. For example, while the discussion of chapter five leading to the invariance properties of fields is invalidated by divergence difficulties, the results are valid when those difficulties are properly treated. On the other hand, it seems that many of the essential results of chapter four cannot be saved except for the case that the charge on the electron is zero.} Chapter seven contains the general theory of the scattering matrix (or more generally the scattering operator), in several forms. Chapter eight discusses the evaluation of the scattering matrix as a power series in the unrenormalized charge. It also includes Furry's theorem, and the relations between scattering matrix elements, cross sections and lifetimes. Chapters nine and ten are devoted to renormalization theory. First, in chapter nine the basic divergences of the scattering matrix occurring in lowest order in the charge are analyzed in detail. Then, in chapter ten, the renormalizability to all orders is proved following the ideas of Dyson and Ward. Chapters eleven through

thirteen apply the general theory in turn to the photon-electron, electron- electron, and photon-photon systems. Chapter fourteen gives the general theory of a system in an external electromagnetic field and chapter fifteen applies this theory to the calculation of Coulomb and Delbrück, pair production and annihilation, bremsstrahlung, the magnetic moment of the electron and the Lamb shift. Chapter sixteen contains an account of a variety of special problems: infra-red divergences, the self stress of the electron, radiation damping in collision processes, and the natural breadth of stationary states. The seven appendices contain respectively, a catalogue of relativistic singular functions, a derivation of the basic properties of the Clifford numbers, the Wick expansion theorem for fields, Feynman's method for doing certain integrals, a limiting relation for the δ-function and the dispersion relation for light scattering. {In the reviewer's opinion this second half of the book beginning with chapter eleven is of such excellence that it alone would be enough to make the book outstanding.}

A. S. Wightman (Princeton, N.J.).

Ginzburg, I. F. On the failure of the weak coupling approximation in two-charge meson theory. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 535–538. (Russian)

Širkov showed [same Dokl. (N.S.) **105** (1955), 972–975; MR **17**, 1033] using the method of the renormalization group that in "two-charge" meson theory, perturbation theory is invalid even when both of the coupling constants are small. ("Two-charge" refers to the two coupling constants of the theory, one in the meson-nucleon interaction and the other in the meson-meson interaction). The present paper establishes the same result independent of the value of the coupling constants, and also determines the asymptotic form of the Green's functions for large momenta in the general case. Since the argument is based on the same assumptions about the existence of certain limits of the Green's functions as were used in the work of Bogoljubov and Širkov [Nuovo Cimento (10) **3** (1956), 845–863], it is subject to the same criticism [MR **17**, 1260]. *A. S. Wightman.*

Marx, G.; and Ziegler, M. A. Relativistic two-body problem in the classical theory of meson field. Acta Phys. Acad. Sci. Hungar. **7** (1957), 125–133. (Russian summary)

The authors consider the following problem. Let a particle at the point r_i and with the velocity $\dot{r}_i$ generate a potential $\varphi(r, t)$ at the point r and time t according to the equation

$$\Delta\varphi - c^{-2}\frac{\partial^2\varphi}{\partial t^2} = 4\pi g\sqrt{(1-\dot{r}_i^2/c^2)}\ \delta(r-r_i).$$

The authors consider φ to be a "classical meson field" for mesons with zero mass. This field acts on another particle at the point r and time t with a force proportional to $-\text{grad}\,\varphi$. The problem of integrating this set of equations for a given set of initial values (positions and velocities of the two particles) is called the "relativistic two body problem" by the authors. They integrate the equations by numerical methods for the special case that the two particles are moving along a straight line towards each other. The retardation of the force implied by the equation displayed above is taken into account. This seems to be the main difference between this paper and previous work [e.g. J. Plebanski and W. Sawicki, Acta Phys. Polon. **14** (1955), 455–470; MR **17**, 1262]. The main result of the paper under review is the conclusion that when the distance between the two particles is so small that the potential φ is bigger than the total initial energy (including the rest masses), i.e. when the mass of each particle computed with the aid of energy conservation should become negative, the two particles reverse their direction of motion even if the original forces are attractive. This is called a "relativistic repulsion". *G. Källén.*

Mills, R. L. Integral equations for meson field theory. Nuovo Cimento (10) **5** (1957), 30–44.

The author develops a very detailed formalism for an approximative computation of S-matrix elements for processes involving only one real nucleon. Neutral, pseudoscalar meson theory is used throughout. The general idea is to express the matrix element of interest as a functional derivative of the vacuum expectation value of the time ordered product of two nucleon fields. (Functional differentation was used in field theory as early as 1934 in a paper by V. Fock [Phys. Z. Sowjetunion **6** (1934), 425–469] and not for the first time in 1951 as the author states in foot note 15.) The twofold vacuum expectation value is determined from a set of integral equations. This set is, in principle, infinite but is made finite with the aid of various approximations (closed loops are neglected, no meson-meson interaction, certain high order kernels are replaced by their value for non interacting fields). The resulting set is renormalized but no practical application of the formalism is given.

G. Källén (Copenhagen).

Królikowski, W.; and Rzewuski, J. On "potentials" in the theory of quantized fields. Nuovo Cimento (10) **3** (1956), 260–275.

An integro-differential equation is obtained for one of the components of the state vector describing a many particle system in quantum field theory. This equation is a generalization of equations obtained from the Tamm-Dancoff method [M. M. Lévy, Phys. Rev. (2) **88** (1952), 72–82; MR **14**, 706]. Using formal methods, the equation is put in the form of a Schrödinger equation; so that it is an ordinary linear eigenvalue equation for the total energy. *J. C. Taylor* (London).

Królikowski, W.; and Rzewuski, J. A new proof of the one-time equation in the theory of bound states. Nuovo Cimento (10) **4** (1956), 1212–1215.

The authors of the present paper recently [see the paper reviewed above] gave a formulation, in terms of one-time Schrödinger type equations, of the many-time equations describing the many-body problem in quantum field theory. The proof of this formulation was carried out by the intermediary of integral equations and it was essential that these equations are inhomogeneous. In the theory of bound states, however, the inhomogeneity of these equations vanishes and thus the proof given in the above quoted papers does not hold. Considering first the inhomogeneous case and then going to the limit is a mathematically unsatisfactory way out. The present paper therefore gives a new proof which applies whether the equations are inhomogeneous or not. The relationship between equations of the old proof and the new derivation are also given. The new proof confirms all the results which were obtained by the old method.

M. Moravcsik (Upton, N.Y.).

Ingraham, Richard; and Ford, Joseph. Boson family from quantized finite-particle Maxwell theory. Phys. Rev. (2) **106** (1957), 1324–1336.

This paper discusses the quantization of the generalization of the Maxwell theory proposed by Ingraham [Phys. Rev. (2) **101** (1956), 1411–1419; MR **17**, 908] in which the underlying five-dimensional space is the space of spacetime spheres centered at x^m ($m=1, 2, 3, 4$) with radius λ. The theory is shown to yield a family of free bosons of non-zero masses and spins 1 and 0. The mass spectrum is shown to be continuous and this may be made discrete by restricting the range of the variable λ to be finite. However, when this is done the mass spectrum obtained does not fit the observed boson mass ratios. An automatic connection between the theory given in this paper and the Pauli-Villars regulator theory is obtained.

A. H. Taub (Urbana, Ill.).

Mandelstam, S. Uniqueness of solutions of the Bethe-Salpeter equation for scattering. Proc. Roy. Soc. London. Ser. A. **237** (1956), 496–512.

The author investigates the asymptotic behaviour at high momentum of the solutions of Bethe-Salpeter integral equation for nucleon-nucleon scattering ("ladder approximation"). The equation is first continued analytically in order to change the metric from hyperbolic to Euclidean. The wave-function is expanded in terms of irreducible representations of the 4-dimensional rotation group, so that the integral equation can be reduced to a set of (coupled) one-dimensional radial second-order differential equations of the conventional type. It is concluded that these equations are soluble for all states of angular-momentum, parity and isotopic spin if and only if $g^2/4\pi<\pi/6$.

A. Salam (London).

Svidzinskii, A. V. Determination of the Green's function in the Bloch-Nordsieck model by functional integration. Soviet Physics. JETP **4** (1957), 179–183.

In the electrodynamic model of Bloch and Nordsieck [Phys. Rev. (2) **52** (1937), 54–59] the Dirac γ matrices are replaced by c-numbers u_α which satisfy $u_0^2-u^2=1$. The Green's function for an electron in a given external field is determined by the equation,

$$\left(iu_\alpha\frac{\partial}{\partial x_\alpha}-eu_\alpha A_\alpha-m\right)G_e(x, x')=-\delta(x-x'),$$

which can be solved by quadratures. The author solves the equation and then uses the result [N. N. Bogoliubov Dokl. Akad. Nauk SSSR (N.S.) **99** (1954), 225–226; MR **16**, 778; P. T. Matthews and A. Salam, Nuovo Cimento (9) **12** (1954), 563–565] that

$$G(x, x')=\frac{\int G_e(x, x')\exp\{i\int L_f\}\delta A}{\int\exp\{\int L_f\}\delta A},$$

where G is the electron Green's function in a quantised field A and L_f is free electromagnetic field Lagrangian, to find a closed expression for $G(x, x')$. After renormalizations which are simply carried out, the Fourier transform of G can be expressed as

$$G(p)=Zi(m_1)^{e^2/\pi}\int_0^\infty dv\exp\{i(|p|-m_1+i\varepsilon)v\}v^{e^2/\pi}.$$

Here $Z=(M/m_1)^{e^2/4\pi}$, $m_1=m+(e^2/2\pi)M$ and $M\to\infty$. The essential simplification arising from Bloch-Nordsieck approximation is that there is no vacuum-polarization. Note that $G(p)$ without renormalization is intrinsically divergent.

A. Salam (London).

Burton, W. K.; and de Borde, A. H. Functional integration in quantum field theory. Nuovo Cimento (10) **4** (1956), 254–269.

This paper is concerned with the problem of determining propagators (vacuum expectation values of products of operators) from Feynman functional integrals [R. P. Feynman, Rev. Mod. Phys. **20** (1948), 367–387; MR **10**, 224] in certain simple cases. A presciption is proposed which separates out vacuum-vacuum expectation values from others. The methods then used are developments of that of Matthews and Salam [Nuovo Cimento (10) **2** (1955), 120–134; MR **17**, 693], which consists in expanding the function of integration in terms of a complete orthogonal set of functions. In particular, the case of a system obeying Fermi statistics is discussed. The problem here is that the Lagrangian vanishes identically if the operators are replaced by c-numbers. The authors' solution is to use, instead, an orthogonal set of functions containing just two mutually anti-commuting quantities.

J. C. Taylor (London).

Candlin, D. J. On sums over trajectories for systems with Fermi statistics. Nuovo Cimento (10) **4** (1956), 231–239.

Tobocman [Nuovo Cimento (10) **3** (1956), 1213–1229; MR **18**, 539] has shown how to derive the Feynman action principle [Rev. Mod. Phys. **20** (1948) 367–387; MR **10**, 224] from the canonical formalism, for simple quantum mechanical systems. This derivation is extended to systems obeying Fermi statistics, for instance a single Fermi oscillator. The difficulty here is that the operators describing the system can only be diagonalized with the help of "anticommuting c-numbers". The author's conclusion is that the derivation can be carried through, but that the Feynman integral cannot be interpreted as a sum over trajectories in any classical sense.

J. C. Taylor (London).

Penfield, R. H.; and Zatzkis, H. Invariance requirements and conservation laws. Acta Phys. Austriaca **10** (1956), 261–266.

It is well-known that there is a difficulty in constructing the Hamiltonian for a gauge-invariant field, such as the electrodynamic field. This difficulty is followed through in the passage from the Lagrangian to the Hamiltonian formalisms, in a rather general case.

J. C. Taylor.

Trautman, A. On the conservation theorems and equations of motion in covariant field theories. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 675–678.

Conservation laws derivable from a Lagrangian are obtained, in the usual way, due to Emmy Noether, under the assumption that the substantial derivative of the field variables $\delta\psi$ is a linear combination of ψ and $\psi_{,\nu}$. The resulting equations are applied to the case of a perfect fluid without pressure to show that the equations of motion appear as integrability conditions.

A. J. Coleman (Toronto, Ont.).

Trautman, A. Killing's equations and conservation theorems. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 679–682.

Continues the paper reviewed above, studying conservation laws associated with variations of the metric. It is first shown that a theory is conformally invariant if the energy-momentum tensor has zero trace. The consequences of Killing's equations, which put restrictions

on the metric, are then explored with applications to flat-space, stationary space and spaces of constant curvature. *A. J. Coleman* (Toronto, Ont.).

Heisenberg, W. Hilbert space II and the "ghost" states of Pauli and Källén. Nuovo Cimento (10) **4** (1956), supplemento, 743–747.

The author considers the role of "ghost" states [G. Källen and W. Pauli, Danske Vid. Selsk. Mat.-Fys. Medd. **30** (1955), no. 7; MR **17**, 927] in his theory of the non-linear wave equation [Heisenberg, Kortel, and Mitter, Z. Naturf. **10a** (1955), 425–446; MR **17**, 330]. In this theory, Hilbert space II was introduced to give negative contributions to cancel the singularities of the Green's functions on the light-cone. Thus Hilbert space II can be viewed as being made up totally from ghosts. Hence the question arises as to whether the unitary of the S-matrix can still be maintained. It is proven in the paper that if one starts with a system of non-interacting particles without ghosts, this system, after scattering, will still not contain ghost states (and hence the S-matrix will be unitary for such a process). The theorem is proven by analysing the possible time dependence of a Hilbert space II state and then showing that such a time dependence can not arise for the above situation. If one were to investigate the wave function during the interaction of the particles, one would need "negative probabilities" to describe the situation, but not in the asymptotic form of the wave function. *R. Arnowitt* (Syracuse, N.Y.).

Wataghin, G. On a non-local relativistic quantum theory of fields. I. Nuovo Cimento (10) **5** (1957), 689–701.

The author considers a new form of non-local field theory. The theory is first formulated for the S-matrix: each propagator entering into a Feynman diagram in momentum space is modified by having an extra invariant form factor multiplying it (which produces convergence). One of the innovations introduced is that the form factor is not a function of k_u^2 (where k_u is the energy-momentum vector carried by the propagator) but rather a function of the variable $I_s^2=k_uu_u-k_u^2$. Here u_u is the velocity of the center of mass of the particles scattering. Thus, in the center of mass (c.m.) frame $I_s^2=\mathbf{k}^2$. The theory is subsequently reformulated in Hamiltonian form. The Hamiltonian is non-local in structure. However, due to the above mentioned dependence of the form factor on I_s^2 in momentum space, the corresponding form factor in coordinate space that appears in the Hamiltonian is local in time in the c.m. system. This has the advantage of preserving the usual Hamiltonian quantization. The theory is also then consistent with the principle of macroscopic causality though, of course, microscopic causality is violated in a small region of radius l, where l is the fundamental length appearing in the form factor.

The physical interpretation given the non-locality is that it is impossible to localize a particle in a region having a smaller radius than l. The author also suggests the existence of an uncertainty in time of order l.
R. Arnowitt (Syracuse, N.Y.).

Tzou Kuo-Hsien. Les états de spin des champs tensoriels et comparaison à ceux des champs de spin maximum entier de la théorie de fusion. C. R. Acad. Sci. Paris **244** (1957), 1886–1888.

The spin states of a tensor field of rank n are examined and shown to be identical to the states in de Broglie's "Theory of fusion" [Théorie générale des particules à spin, 2nd ed., Gauthiers-Villars, Paris, 1954] for a field built up from $2n$ half integral spin fields. The theorem is established by first decomposing a vector field into its spin one and spin zero parts and then describing an nth rank tensor as a product of n vectors. Similarly the de Broglie field that carries a maximum spin of n ($n=$ integer) can be viewed as the product of $2n$ spin $\frac{1}{2}$ systems. Since the product of two spin $\frac{1}{2}$ systems can be decomposed into a spin 1 plus a spin 0 part, the theorem follows.
R. Arnowitt (Syracuse, N.Y.).

Grigoriev, V. I. Generalized method for calculating damping in relativistic quantum field theory. Soviet Physics. JETP **3** (1956), 691–696.

A method is discussed for solving quantum field theory problems which leads to a generalized theory of radiation damping. The starting point is the Tomanaga-Schwinger equation [Tomonaga, Progr. Theoret. Phys. **1** (1946), 27–42; Schwinger, Phys. Rev. (2) **75** (1949), 651–679; MR **10**, 663] for the U-matrix in the interaction representation:

$$i\delta u[\sigma]/\delta\sigma(x)=H(x)u[\sigma].$$

U is then decomposed into sub parts, $U=\sum U_n$, where the U_n govern the transition where i photons are absorbed j photons are emitted, etc. A set of coupled equations for the U_n are then set down of the form

$$i\delta u_n/\delta\sigma=\sum_{n'} H_{nn'}u_{n'}.$$

These equations are to be solved under the boundary conditions $U_0(-\infty)=1$, $U_n(-\infty)=0$ $(n\neq 0)$ as U_0 governs the time development of the initial state. These equations are first iterated an infinite number of times. In the equation for U_n, one will find, in general, terms on the right depending on U_0, U_n, and $U_{n'}$ $(n'\neq n)$. The last set of terms is neglected (an approximation that is presumably valid in the weak coupling limit) leaving only coupled equations between U_0 and U_n to solve. Solutions are then found of the form perturbation theory times a constant and exhibit the usual damping structure. Renormalization problems are not discussed though the author leaves the impression that renormalization can be achieved. *R. Arnowitt* (Syracuse, N.Y.).

Green, H. S. Renormalization with pseudo-vector coupling. Nuclear Phys. **1** (1956), 360–362.

This note is divided into two parts. The first part discusses the results of Arnowitt and Deser [Phys. Rev. (2) **100** (1955), 349–361; MR **17**, 926] on the renormalization of pseudovector coupling meson theory in terms of the author's previous formulation of field theory [H. S. Green, ibid. **95** (1954), 548–556; MR **16**, 318]. The second part deals with an approximate integral equation for the vertex operator for pseudovector coupling analogous to the one previously treated by Edwards [ibid. **90** (1953), 284–291; MR **15**, 83] for pseudoscalar coupling. *R. Arnowitt.*

van Kampen, N. G. Can the S-matrix be generated from its lowest-order term? Physica **23** (1957), 157–163.

The paper clarifies by means of simple examples the true significance of the Kramers-Kronig relations for S-matrix theory and stresses an important point which has not always been clearly understood: the question forming the title of the paper must be answered in the negative. If the S-matrix is written as a perturbation expansion, formal application of this expansion in the unitary condition and the Kramers-Kronig relations would determine

in succession all terms of S from the lowest order on. The fallacy of this argument lies in the lack of convergence of the series involved. This is illustrated in the case of a scalar field of mass zero scattered by a harmonic oscillator, in the monopole (s-wave) approximation. The case of light propagation through a dispersive medium is also considered. In the latter context the author stressed how confusing and unsatisfactory it is to use the name dispersion relation instead of Kramers-Kronig relation.

L. Van Hove (Utrecht).

McVoy, Kirk; and Steinwedel, Helmut. Principal axis transformation for a free nucleon coupled to an uncharged scalar meson field. Nuclear Phys. 1 (1956), 164–179.

A non-relativistic Schrödinger particle (nucleon, position R) is coupled with a neutral Klein-Gordon field φ (mesons) through a scalar interaction $g\varphi(R)$. If this is expanded in powers of R, the zeroth term $g\varphi(0)$ is absorbed in a constant, infinite static self energy. The first-order term, together with the uncoupled Hamiltonian forms a quadratic expression in R, φ, and their canonical conjugates. The second-order term gives rise to mass renormalization and higher terms are neglected.

The quadratic Hamiltonian can be transformed to principal axes by two successive canonical transformations. This amounts to a solution of the Schrödinger equation for small R but to all powers of g. There are three types of eigenfunctions. (i) The nucleon moves with constant velocity R, accompanied by a static meson cloud. (ii) Oscillatory motions describing the scattering of incident mesons by the nucleon. (iii) A self-accelerating solution similar to the one found for electrons in interaction with the electromagnetic field.

N. G. van Kampen (Utrecht).

Parasyuk, O. S. Multiplication of causal functions for non-coincident arguments. Izv. Akad. Nauk SSSR. Ser. Mat. 20 (1956), 843–852. (Russian)

A theory is constructed for the multiplication of the causal functions of quantum field theory, for non-coincident arguments. It is based on the theory of generalized functions of S. L. Sobolev [Mat. Sb. N.S. 1(43) (1936), 39–72] and L. Schwartz [Théorie des distributions, t. I, II, Hermann, Paris, 1950, 1951; MR 12, 31, 833].

N. Rosen (Haifa).

Wrzecionko, J. Character of interaction potential between electron and photon. Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 669–674.

An electron-photon interaction potential is obtained from a consideration of electron-photon scattering based on the Tamm-Dancoff method in the Dyson approximation [F. J. Dyson et al., Phys. Rev. (2) 95 (1954), 1644–1658].

N. Rosen (Haifa).

Livshits, M. S. The application of non-self-adjoint operators to scattering theory. Soviet Physics. JETP 4 (1957), 91–98.

The problem discussed here is the case of elastic scattering in which an incident particle a and a target nucleus X combine to form a compound nucleus C which decays back into a and X. The properties of the scattering matrix are determined from the ansatz that the wave function of the particle a can be described as a vector with two components φ and ψ as follows:

$$\varphi(n)=\begin{cases}\text{const}[e^{-ik(n-R)}-S_0(w)e^{ik(n-R)}] & (n>R)\\ 0 & (n<R)\end{cases}$$

$$\psi=\begin{cases}0 & (n>R)\\ \sum C_j\psi_j & (n<R)\end{cases}$$

where ψ_j is a complete set of orthonormal functions describing the compound nucleus, $W=h^2/2\mu$, μ=particle mass, R=channel radius. The Hamiltonian operator can be then written as a two by two matrix where the complex diagonal elements couple the outside wave function φ and the inside wave function ψ. A formal solution for the scattering matrix S_0 is found and further properties based on formal properties such as the Breit-Wigner formula of the solution found. *H. Feshbach* (Cambridge, Mass.).

Akhieser, A. I.; and Sitenko, A. G. Diffractional scattering of fast deuterons by nuclei. Phys. Rev. (2) 106 (1957), 1236–1246.

The authors show how the usual treatment of the diffraction of light by an absolutely black sphere can be generalized to obtain the diffractional scattering cross section of two weakly bound particles, such as deuterons, by absolutely black nuclei. In this way they obtain the cross sections for the diffractional scattering of fast deuterons by nuclei both with as well as without the disintegration of deuterons. They also consider the effect of the Coulomb interaction on the diffractional scattering. Some of their results are identical with earlier results obtained by others by means of the perturbation theory.

S. N. Gupta (Detroit, Mich.).

Salecker, H. On gauge invariance in quantum electrodynamics. Nuovo Cimento (10) 4 (1956), supplemento, 733–737.

The author reconsiders the old worry as to why calculations of the photon's self-mass in quantum electrodynamics sometimes give non-vanishing results despite the fact that the theory is gauge invariant. The solution he suggests is as follows: let the bare photon have a rest mass m_b (i.e. the bare photon will be a neutral vector meson). When the interaction is turned on, a dynamical mass m_i will arise. The condition to be imposed is that the observed mass, m_0, be zero, i.e. $m_b+m_i=m_0=0$. Thus the renormalized theory will be gauge invariant as will all observable quanties. The theory is also relativistically invariant at all stages. *R. Arnowitt* (Syracuse, N.Y.).

Ramakrishnan, Alladi; and Srinivasan, S. K. A new approach to the cascade theory. Proc. Indian Acad. Sci. Sect. A. 44 (1956), 263–273.

In the current treatments of cosmic ray showers, one considers the function

$$\pi^i(n_1, n_2, \cdots, n_m; E_1, E_2, \cdots, E_m|E_0; t)$$

which defines the probability that there exists n_j particles of type j each of which has an energy greater than E_j ($j=1, \cdots, m$) at depth t given that the cascade was initiated by a particle of type i with an energy E_0. In the "new approach" described in this paper a function $\Pi^i(n_1, n_2, \cdots, n_m; E_1, E_2, \cdots, E_m|E_0; t)$ is considered which defines the probability that n_j particles of type j each with energy greater than E_j at the time of its production are produced between 0 and t. The manner in which one may write down equations for the various moments of Π^i is described. Thus if $f_1{}^i(E|E_0, t)dE$ and

$g_1^t(E|E_0, t)dE$ are the probabilities that an electron and photon (respectively) with energies in the interval $(E, E+dE)$ occur at depth t, then the probabilities $F_1^t(E|E_0, t)dEdt$ and $G_1^t(E|E_0, t)dEdt$, that an electron and photon (respectively) with energies in the interval $(E, E+dE)$ are produced between t and $t+dt$, are given by

$$F_1^t(E|E_0; t)=2\int_E^{E_0} g_1^t(E'|E_0, t)\rho(E|E')dE',$$

$$G_1^t(E|E_0; t)=\int_E^{E_0} f_1^t(E'|E_0, t)R(E'-E|E')dE',$$

where $\rho(E|E')$ is the probability per unit thickness that a photon of energy E splits up into a pair of electrons one of which has the energy E' and $R(E'|E)$ is the related probability that an electron of energy E radiates a quantum of radiation and becomes one of energy E'. It is indicated how one can write similar relations for the higher order functions such as

$$F_k^t(E_1, E_2, \cdots, E_k|E_0; t_1, t_2, \cdots, t_k)$$

which governs the probability of occurrence of electrons with energies $E_1, E_2, \cdots, E_k$ at $t_1, t_2, \cdots, t_k$, respectively. It is further shown how the various resulting integral relations can be solved by taking their Mellin transforms.
S. Chandrasekhar (Williams Bay, Wis.).

Srinivasan, S. K.; and Ranganathan, N. R. Numerical calculations on the new approach to the cascade theory. I. Proc. Indian Acad. Sci. Sect. A. **45** (1957), 69–73.

Before physical conclusions can be drawn from the solutions obtained in the paper reviewed above it is necessary that their inverse Mellin transform be obtained. In this paper the required transformation is carried out, approximately, by a method of steepest descent. Some numerical results are presented. *S. Chandrasekhar.*

Corben, H. C. New approach to the quantum theory of the electron. Phys. Rev. (2) **104** (1956), 1179–1185.

The author's summary reads in part: "Expectation operators and total operators are defined for a quantized field theory in a manner analogous to that in which expectation values are defined in unquantized field theory. From the usual anticommutation relations of the ψ field it is possible to determine the commutation relations between such operators. The quantized theory may then be developed in some detail without recourse to plane wave expansions. Indeed, it is shown that at least one divergence of the standard method — the vacuum charge expectation value — arises from the use of such an expansion." *A. J. Coleman* (Toronto, Ont.).

Aržanyh, I. S. Representations of the meson field by retarded potentials. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 953–956. (Russian)

The meson field satisfies A. Proca's equations [J. Phys. Radium (7) **7** (1936), 247–353]:

$$\operatorname{rot} H=\frac{1}{c}\frac{\partial E}{\partial t}-k^2A, \quad \operatorname{div} E=-k^2\varphi,$$

$$E=-\operatorname{grad}\varphi-\frac{1}{c}\frac{\partial A}{\partial t}, \quad H=\operatorname{rot} A.$$

The purpose of the present note is to represent, by means of retarded potentials corresponding to Klein's operator $\nabla^2-k^2-c^{-2}\partial/\partial t$, the vectors E and H and the potentials φ and A. The method employed is the same as that of the author in previous papers concerning related operators in the theory of electricity, hydrodynamics, and the theory of elasticity [Dokl. Akad. Nauk SSSR (N.S.) **85** (1952), 55–58; **100** (1955), 1053–1056; Uspehi Mat. Nauk (N.S.) **10** (1955), no. 1(63), 202–204; MR **14**, 46; **16**, 1181]. If $k=0$, then certain of the formulas given here reduce to the corresponding formulae for Maxwell electrodynamics given in the last two papers mentioned, and a second set of formulae for E and H reduce to Kirchhoff's formula. Integro-differential equations, with retarded arguments, are also constructed for the solutions of two typical boundary value problems for Klein's equation.
J. B. Diaz (College Park, Md.).

Hiida, K.; and Sawamura, M. Some relations among Green's functions. Nuovo Cimento (10) **5** (1957), 896–906.

The authors consider pseudoscalar meson theory with pseudoscalar interaction. The vacuum expectation value of a product of three fields ($\varphi(x)$=the meson field, $\psi(x)$=the nucleon field) can, in momentum space, be written in terms of a function $G(p, k)$

$$\langle 0|\psi(x)\bar{\psi}(y\chi(z)|0\rangle=\iint dpdkG(p, k)e^{ip(x-y)+ik(y-z)},$$

which is different from zero only when both p and k are positive, time-like vectors. This follows from very general assumptions about the mass spectrum of the theory. The authors use this fact together with some general relations following from the invariance properties of the theory to derive a "spectral representation" for the time ordered product of the operators mentioned above. The weight functions appearing in this representation are essentially given by the function G above. It is the reviewer's opinion that although the results of the authors are correct as far as they are stated, they are nevertheless incomplete in one important respect. For the time ordered product to be an invariant object it is necessary that the fields involved commute (anticommute) for space like separations. This point is by-passed by the authors because they perform all computations in a very special frame of reference. The results that are obtained in this way are correct only if the fields have the correct commutative properties. However, these formulae do not permit the use of arbitrary weight functions and the result would, in that case, have no bearing whatever on the relation between the time ordered and non time ordered products. The reviewer thinks that it is one of the important unsolved problems for the moment to find out the exact restrictions on the function G above that follow from the commutativety of the fields for spce like separations. This problem is not discussed by the authors. *G. Källén.*

Kortel, F. On some solutions of Gürsey's conformal-invariant spinor wave equation. Nuovo Cimento (10) **4** (1956), 210–215.

This paper gives some solutions of the general equation

$$\gamma_\mu \frac{\partial\psi}{\partial x_\mu}+l^{3n-1}(\bar{\psi}\psi)^n\psi=0,$$

$l>0$, n real, and also some additional solutions for the special case of $n=\frac{1}{3}$, which is the equation proposed by Gürsey (Nuovo Cimento (10)3(1956), 988–1006; MR **18**, 173) as a possible basis for a unitary description of elementary particles. The solutions are sought in the following form, suggested by Heisenberg [Z. Naturf. **9a** (1954), 292–303; MR **15**, 915]

$$\psi=(x_\mu\gamma_\mu\chi(s)+\varphi(s))a,$$

with a an arbitrary constant spinor, $\chi(s)$ and $\varphi(s)$ real functions, and

$$s=-x_\mu^2=t^2-r^2.$$

The procedure is to find a pair of differential equations for the χ and φ, and solve these. Some of the solutions are quite simple, such as powers of s, others quite complicated, involving Abelian functions. A discussion of the physical implications of these solutions is promised to follow in another paper. *M. J. Moravcsik* (Upton, N.Y.).

Fubini, S.; and Thirring, W. E. Theory of p-wave pion-nucleon interaction. Phys. Rev. (2) **105** (1957), 1382–1387.

Assuming a static meson theory with a rather general interaction Hamiltonian with linear coupling, a number of inequalities between certain renormalization constants, r_i, are obtained. The derivation is based on the requirement, that the probabilities for the meson cloud surrounding a physical nucleon to be in certain states must be non-negative. From the Low equation [Phys. Rev. (2) **97** (1955), 1392–1398] it follows further that for the p-wave pion-nucleon interaction certain equalities hold which relate the renormalized coupling constant f^2, the constants r_i, the expectation value of the self-energy, E, the total charge, and other observables, to certain integrals over the scattering phase shifts. Using the value $f^2=0.08$ previously established from the increase of the phase shift at low energies, the above relations can be exploited to yield semi-empirical constants in the special case of the Chew model Hamiltonian by comparing them with other low energy experimental data. The authors obtain rather definite values for the r_i, the unrenormalized coupling constant, and the shape parameter of a Yukawa source function for the nucleon. However, it is somewhat questionable whether these semi-empirical results could consistently follow from the theory if it could be carried through completely. *F. Rohrlich* (Iowa City, Ia.).

Tanaka, Sho. The composite model for new unstable particles. I, II. Progr. Theoret. Phys. **16** (1956), 625–630, 631–640.

The new unstable particles are considered to be composed of nucleons, Λ-particles and their anti-particles, as proposed by S. Sakata [same journal **16** (1956), 686–688], but with the modification that two kinds of Λ-particles are assumed. On the basis of the proposed model, calculations for the decay of some of these particles are carried out. *N. Rosen* (Haifa).

d'Espagnat, B.; et Prentki, J. Formulation mathématique du modèle de Gell-Mann. Nuclear Phys. **1** (1956), 33–53.

Assuming that isotopic-spin space is 3-dimensional, and that the Lagrangian for strong interactions is isotopic-rotation invariant, an axiomatic formulation of Gell-Mann's [Phys. Rev. (2) **92** (1953), 833] ideas for classification of strange particles is attempted. The starting point is Gell-Mann's proposal that $N=(p, n)$, $\Xi=(\Xi^0, \Xi^-)$, and $k=(k^+, k^0)$ particles are iso-spinors, Λ^0 is iso-scalar and $\mathbf{\Sigma}=(\Sigma_1, \Sigma_2, \Sigma_3)$ and $\mathbf{\Pi}$-mesons $(=\Pi_1, \Pi_2, \Pi_3)$ are iso-vectors. The rotation-invariant 3-field Lagrangian contains 8 possible couplings between these particles if charge and nucleon-number are conserved. The Lagrangian is found to be additionally invariant for the gauge-transformations $\psi\to\psi e^{i\alpha}$, where ψ is wave-function for an iso-spinor. This invariance results in the conservation of iso-fermions, i.e. U is the number of iso-fermions — number of anti-iso-fermions is constant. In this context, since charge-conservation requires a $\Xi\Lambda\tau_2 k^*$ coupling in contrast to $\bar{N}\Lambda k$ coupling, the Ξ particle must be treated as an anti-iso-fermion with $U=-1$. With this assignment of U for the Ξ particle, the gauge-transformation above can be generalised to read $\psi\to\psi e^{iU\alpha}$, where ψ refers to the isotopic multiplets ($U=1$ for N, K particles, -1 for Ξ, and 0 for Λ and Σ hyperons). The authors remark that if $\alpha=\frac{1}{2}\pi$, the gauge-transformation acquires the significance of an isotopic-reflection, provided N and Ξ iso-spinors transform differently (spinors of first and second kind), Λ is scalar and Σ and Π are considered as iso-pseudo-vectors. The quantum-number U is simply connected with Gell-Mann's "strangeness" S ($U=S+N$, where N is nucleon-number), while its relation with charge is given by $Q=\frac{1}{2}U+I_3$. *A. Salam* (London).

Racah, Giulio. Remarques sur une formulation mathématique du modèle de Gell-Mann. Nuclear Phys. **1** (1956), 302–303.

Murai, Yasuhisa. Inversion in isobaric spin space. Nuclear Phys. **1** (1956), 657–659.

Racah and Murai independently elaborate the role of isotopic-reflections for the interpretation of the quantum-number U introduced by D'Espagnat and Prentki [see the paper reviewed above]. Since $U=2(Q-I_3)$, the gauge-transformation $\psi\to e^{iU\alpha}\psi$ is merely a combination of the charge-conserving gauge $\psi\to e^{iQ\theta}\psi$ and a rotation in isotopic space $\psi\to e^{iI_3\phi}\psi$. The fact that for the particular angle $\alpha=\frac{1}{2}\pi$, it can be interpreted as a reflection operation in isotopic-space is a "gift" in the formalism. *A. Salam.*

Petiau, Gérard. Sur la détermination des fonctions d'ondes du corpuscule de spin $\hbar$ en interaction avec un champ magnétique ou électrique constant. J. Phys. Radium (8) **17** (1956), 956–964.

This article solves the single-particle wave equation for a charged vector meson interacting with a constant electric or magnetic field. The wave-equation is written in essentially the Duffin-Kemner form. Using De Broglie's [Une nouvelle conception de la lumière, Hermann, Paris, 1934; Nouvelles recherches sur la lumière, ibid, 1936] representation of β matrices in terms of systems of two Dirac matrices, the author is able to make direct connection with corresponding solutions of the Dirac equation. *A. Salam* (London).

Lee, T. D.; Oehme, Reinhard; and Yang, C. N. Remarks on possible noninvariance under time reversal and charge conjugation. Phys. Rev. (2) **106** (1957), 340–345.

For the interrelations of (P)arity, (C)harge-conjugation, (T)ime-reversal conservation, the CPT theorem implies that, if one of these be not conserved, at least one other is not conserved. It is proved that if a particle A decays through the weak part H_w of a proper-Lorentz-invariant Hamiltonian H, the strong part being invariant under C, P, T, then, if the particle and its antiparticle $\bar{A}$ do not decay into the same final products to the lowest order in H_w the lifetimes of A, $\bar{A}$ are the same even if H_w is not invariant under C: if, in addition, $[H, C]=0$ and if the decay products are free particles, then, to the lowest order in H_w, there is no interference between parity-conserving and parity-non-conserving parts of H in the decay of A in experiments measuring $\boldsymbol{\sigma}\cdot\mathbf{p}$. The decay of K_0 and $\bar{K}_0$ is discussed in some detail. *C. Strachan.*

Okabayashi, Takao; and Sato, Shigeo. A method of renormalization for unstable particles. Progr. Theoret. Phys. **17** (1957), 30–42.

Lee's model [Phys. Rev. (2) **95** (1954), 1329–1334; MR **16**, 317] is discussed for unstable particles. The interaction term in the Hamiltonian contains $\delta m a_v^+ a_v$ referring to the V fermion field and cancelling change of mass due to $V \rightleftarrows N+\theta$, ($\theta$ a boson field). To determine δm for stable particles the physical requirement is that, for a bare particle initial state, the state function has time dependence $\exp(-imt)$ as $t\to\infty$, giving a pole at $E=m$ for the Fourier Transform of the Green's Function. For unstable particles it is shown that no solution other than $\gamma=0$ exists for a pole at $E=m_v-i\gamma/2$. It is now assumed that the energy spectrum of particles from V decay must have a maximum at $E=m_v$. For coupling-constant renormalization $Z^{\frac{1}{2}}$ is defined as the probability amplitude of the bare V state for an eigenstate of the total Hamiltonian in which the θ field is a standing wave. The time dependence of the instability is more complicated than $\exp(-\gamma t/2)$.

C. Strachan (Aberdeen).

Bopp, Fritz. Eng korrespondierendes klassisches Modell eines quantenmechanischen Elektrons. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1956**, 1–13 (1957).

From the solutions of a one-dimensional Schrödinger-like equation for an electron in a given field may be constructed the solution of the Liouville equation for a statistical ensemble of one-dimensional classical electrons, the one-dimensional radius of the electron being an integral of the equations of motion involving hidden internal parameters.

A further mathematical assumption leads to the Wigner [Phys. Rev. (2) **40** (1932), 749–759] equation for the distribution function for macrovariables. The Liouville distribution function is next replaced by an error-function mean value. This model is claimed to have closer correspondence to quantum mechanics than that of the point-electron. *C. Strachan* (Aberdeen).

Budini, P.; and Fonda, L. Non-local models of pion-nucleon, pion-hyperon interactions. Nuovo Cimento (10) **5** (1957), 666–683.

A meson theory is proposed in which pions interact with nucleons and hyperons, not directly, but by way of an intermediate, parity-doublet K-meson field. In a linear approximation, in which the K-meson field is eliminated, the theory is equivalent to a non-local theory of pion-nucleon interactions. It is argued that such a non-local form of interaction is indicated by experiment.

An exactly soluble model theory of the same general type, gives a repulsive cone in the nucleon-nucleon potential. *J. C. Taylor* (London).

Sudakov, V. V. Consequences of the renormalizability of quantum electrodynamics and meson theory. Soviet Physics. JETP **4** (1957), 616–617.

The asymptotic expressions for the Green functions and vertex part in meson theory due to Abrikosov, Galanin and Khalatnikov [Dokl. Akad. Nauk SSSR (N.S.) **97** (1954), 793–796; MR **16**, 317] are re-derived as consequences of the renormalizability of the theory, using the methods of Gell-Mann and Low [Phys. Rev. (2) **95** (1954), 1300–1312; MR **16**, 315]. *J. C. Taylor* (London).

Rayski, J. Bilocal field theories and their experimental tests. II. Nuovo Cimento (10) **5** (1957), 872–885.

This paper is a continuation of Nuovo Cimento (10) **4** (1956), 1231–1241. Wave equations for families of elementary particles are written down which involve in addition to the usual space-time, spin, and isotopic spin variables, a three-vector isotopic variable. For a particle of mass, this three-vector lies on a sphere determined by $\mathbf{u}^2=l^4m^2$, where l is a fundamental length. Each of these equations yields a mass spectrum. For example, the equation

$$(ip_\mu p_\mu+\mathbf{k}\cdot\mathbf{k})\psi=k^2\psi,$$

where p_μ is the particle four-momentum, $k_j=-i\partial/\partial u_j$ and $k=\frac{1}{2}[u^{-1}\mathbf{u}\cdot\mathbf{k}+\mathbf{k}\cdot\mathbf{u}\cdot u^{-1}]$ determines the spectrum

$$m_a=[a+\tfrac{1}{2}]^{\frac{1}{2}}l^{-1}\quad (a=\tfrac{1}{2},\tfrac{3}{2},\tfrac{5}{2},\cdots).$$

There turns out to be one other natural possibility for particles of integer spin and three others for particles of half-odd integer spin with analogous simple formulae for the mass spectra. When l is adjusted seperately for the cases of integer and half-odd integer spin it turns out to be possible to assign all known elementary particles of mass greater than that of the π-mesons (including some rare events) to the above spectra with an accuracy of a few percent. The discrete variable a occurring in the formula for the mass is definable in terms of the strangeness or attribute. The remainder of the paper is devoted to a discussion of possible physical interpretations of the additional variable $\mathbf{u}$. The notions isoparticle and isotor are introduced. An isotor is a pair of rotators rotating in opposite senses so their resultant angular momentum is zero. An isoparticle is a body with an internal structure which is an isotor. Restmass, charge, attribute, and isospin are to be interpreted as different manifestations of internal motion. Two isoparticles interact only when their isotors are oriented in the same direction. It is suggested that a correct theory of elementary particles will involve fields depending on three kinds of variables: space-time coordinates, internal variables describing the direction of a rotator, and variables describing the degrees of freedom of an isotor. *A. S. Wightman* (Princeton, N.J.).

See also: Maslov, p. 155; Bergmann, Goldberg, Janis and Newman, p. 190; Brittin, p. 190; Herman, p. 199; Zubarev, p. 209; Schützer, p. 210; Ritchie, p. 210; Gosar, p. 210; Gürsey, p. 224; Costa de Beauregard, p. 225; Infeld and Plebański, p. 225.

Relativity

Gürsey, F. Relativistic kinematics of a classical point particle in spinor form. Nuovo Cimento (10) **5** (1957), 784–809.

The author studies the geometry of a world-line in Minkowski space-time by means of the two-component two index spinors associated with the vector tangent to the world-line and a set of three vectors orthogonal to this vector and to each other. Special world lines are studied in detail. The author has previously used a special Galilean coordinate system and a special spin coordinate system to set up a correspondence between two-component two index spinors and four component spinors. The author uses this correspondence to relate the world-line of a free spinning particle as described by Weyssenhoff [Acta Phys. Polon. **9** (1947), 46–53; MR **14**, 214] to the spinor equations recently postulated by Proca [J. Phys. Radium (8) **15** (1954), 65–72; Nuovo Cimento (10) **2** (1955), 962–971; MR **15**, 836; **17**, 558]. *A. H. Taub.*

Infeld, L.; and Plebański, J. Expansion of singular functions associated with the Klein-Gordon equation. Acta Phys. Polon. 15 (1956), 207–248. (Russian summary)

The Green's functions associated with the Klein-Gordon equation are discussed by use of expansions in powers of c^{-1} and c respectively. In each case it proves necessary to make a different assumption concerning the power in c of the mass term. This choice is such as to make this term homogeneous with the time-derivative one in the c^{-1} treatment (which then formally resembles a static Yukawa problem), and homogeneous with the space derivatives in the c expansion, with corresponding analogy to the harmonic oscillator problem. Various expansions for several of the singular functions are tabulated. *S. Deser.*

Štraskevič, A. M. New physico-engineering methods for modelling and computation of trajectories of relativistic charged particles in electrostatic fields. Dokl. Akad. Nauk SSSR (N.S.) 108 (1956), 440–443. (Russian)

The idea is that the motion of a particle of variable mass in a field with given potential ϕ is equivalent to a motion of a particle of constant mass in a field determined by a modified potential ϕ^*. A table is given in which results of computation based on this idea are compared with results obtained by other methods.
G. Y. Rainich (Ann Arbor, Mich.).

Fokker, A. D. Accelerated spherical light wave clocks in chronogeometry. Nederl. Akad. Wetensch. Proc. Ser. B. 59 (1956), 451–454.

As a model clock in special relativity the author takes "··· a closed perfectly rigid, perfectly reflecting spherical mirror, at the centre of which a light pulse has been started. The test for the perfection and rigidity of the sphere is that the light pulse after reflections will ever and ever again concentrate in the centre, and that this focussing will not be disturbed by rotating the body of the spherical mirror." The purpose of the paper is to show that this focussing property is not disturbed by uniform acceleration of the clock. *A. J. Coleman* (Toronto, Ont.).

Fokker, A. D. Accelerating spherical light wave clocks in chronogeometry. Physica 22 (1956), 1279–1282.

Abbreviation of the paper reviewed above.
A. J. Coleman (Toronto, Ont.).

Costa de Beauregard, O. Covariance relativiste à la base de la mécanique quantique. II. J. Phys. Radium (8) 17 (1956), 872–875.

[For part I see same J. (8) 15 (1954), 810–816; MR 16, 984.] It is well-known that the solution of Dirac's equation for a free particle can be expressed in terms of the solution of Gordon's equation. Umezawa and Visconti [Nuclear Physics, (1956) 1, 20–32] have defined a projection operator which extends this result to particles of arbitrary spin satisfying a linear first order equation. This operator is here used by the author to simplify the formulas of two previous papers [C. R. Acad. Sci. Paris 242 (1956), 1581–1583, 1692–1694; MR 17, 929]. He also shows that the orthogonality of two wave functions, usually defined by means of an integral on a three dimensional space-like hypersurface, can be expressed by means of a four dimensional integral. *A. J. Coleman* (Toronto, Ont.).

Infeld, L.; and Plebański, J. On modified Dirac δ-functions. Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 687–691.

In their introduction the authors write: "The present note is written by physicists and we hope it will be of use to physicists. Its aim is to show how, by narrowing the concept of Dirac's δ-functions, we can enrich them with properties which are useful in applications." They contrast the axiomatic, Fourier and realistic methods of defining the Dirac function. The latter involves a limit of a sequence of ordinary functions together with an unjustified interchange of the order of taking the limit and integrating. The authors define a modified function $\hat{\delta}$ which has the usual properties including $\int \hat{\delta}(x)f(x)dx=f(0)$ if f is continuous at 0, but in addition for the particular functions $|x|^{-p}$, where p is an integer $0<p\leq k$,

$$\int |x|^{-p}\hat{\delta}(x)dx=0.$$

The definition is effected realistically by exhibiting a set of functions $\delta_n(x)$ such that $\hat{\delta}$, defined by

$$\lim_{n\to\infty}\int \delta_n f=\int \hat{\delta} f,$$

enjoys the above properties. *A. J. Coleman.*

Infeld, L.; and Plebański, J. On a covariant formulation of the equations of motion. Bull. Acad. Polon. Sci. Cl. III 4 (1956), 757–762.

The matter tensor of Einstein's equations is expressed as a sum of point singularities of the modified δ-function defined in the paper reviewed above. It is shown that the field equations constrain the singularities to move on geodesics. The reason for the definition of $\hat{\delta}$ appears in this paper. It eliminates the self-action of the point particles which give rise to pole singularities. One would therefore expect that correct results "proved" with the use of $\hat{\delta}$ could also be obtained by Hadamard's method of "parties finies". *A. J. Coleman* (Toronto, Ont.).

Infeld, L.; and Plebański, J. On the "dipole procedure" in general relativity theory. Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 763–767.

The methods of the two papers reviewed above are used to improve the famous paper of Einstein and Infeld [Canad. J. Math. 1 (1949), 209–241; MR 11, 59] in which the idea first occurred of obtaining the equations of motion of point particles as constraints imposed by the field equations on the singularities of the field. The present paper attempts to overcome two alleged defects of this earlier work: 1) its procedure for obtaining the actual motion of the particles was not covariant 2) there was some doubt about the validity of the approximation procedure used.

A tensor D^m is introduced covariantly into the equations in such a way as to leave the singularities free to have any motion. The resulting equations are solved by an approximation procedure in terms of increasing powers of c^{-1}. The authors then argue that the assumption of geodetic motion of the singularities (required by the previous paper) forces D^m to vanish and reduces the field equations to Einstein's so that they attain the field and the motion of the particles to any order of approximation. In the lowest order, they assert, the motion is Newtonian. It is this last point, as to whether, or in what sense, Einstein's field equations force Newtonian motion on the singularities in first approximation, that there has been much debate since the original paper of Einstein and Infeld. [Reviewer's comment: The paper is elegant and interestint but fails to come to grips with the problem of coordinate conditions which has been much discussed in

recent years. This is avoided by the facile assumption in the paragraph headed "Third" in section 2, that the geodetic equations imply $D^m=0$. In fact, their equation 3.7 follows from 3.6 only by implicit assumptions as to the behaviour of D^m at infinity. But where is infinity in the space of general relativity? The paper does seem to prove that Newtonian motion of the singularities in first approximation in the small is consistent with the field equations. The fundamental question would be answered if it were known that the solution is unique. This might follow from showing that the vanishing of the sum of the extra terms in 2.1 implied the vanishing of the left side of equation 2.2. of the paper]. *A. J. Coleman.*

Infeld, L.; and Plebański, J. On a further modification of Dirac's δ-functions. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 51–54, VI. (Russian summary).

An extension of an earlier paper reviewed third above treating other continuous functions which, in the limit, reduce to the three-dimensional Dirac delta function. *P. M. Morse* (Cambridge, Mass.).

Infeld, Leopold. On the equations of motion. Schr. Forschungsinst. Math. **1** (1957), 202–209.

L'auteur rappelle brièvement et clairement les principes de la relativité générale et l'historique des équations du mouvement. Il indique les principales idées qui ont permis d'obtenir ces équations par la méthode des singularités [A. Einstein, U. Infeld et B. Hoffmann, Ann. of Math. (2) **39** (1938), 65–100; A. Einstein et U. Infeld, Canad. J. Math. **1** (1949), 209–241; MR **11**, 59] et la méthode du tenseur d'impulsion énergie [U. Infeld, Acta Phys. Polon. **13** (1954), 187–204; MR **16**, 531; cf. aussi Fock, Acad. Sci. U.S.S.R. J. Phys. **1** (1939), 81–116; MR **1**, 183; F. Hennequin, Thèse, Paris, 1956]. *Y. Fourès-Bruhat.*

Hoffmann, Banesh. General relativistic red shift and the artificial satellite. Phys. Rev. (2) **106** (1957), 358–359.

This paper supplements the work of S. F. Singer [Phys. Rev. (2) **104** (1956), 11–14; MR **18**, 782] by taking into account the rotation of the earth (thus changing the motion of the observer in the Schwarzschild field) and the earth's spheroidal shape (which is allowed for by changing the term of the type $1/r$ in the Schwarzschild line element into the form which, in Newtonian theory, gives the potential of an oblate spheroid). The two resulting corrections to the red shift are of the order 10^{-12} and pull in opposite directions. *J. L. Synge* (Dublin).

Takasu, Tsurusaburo. A fact, which is unfavorable to the theory of general relativity of A. Einstein. Proc. Japan Acad. **32** (1956), 535–538.

The author draws the attention to the fact that in Einstein's theory [The meaning of relativity, 4th ed., Princeton, 1953; MR **14**, 805] the momentum vector is not tangential to the conjectured path of the free particle. He maintains that this leads to a self-contradiction. *J. A. Schouten* (Epe).

Takeno, Hyôitirô. Some wave solutions of Einstein's generalized theory of gravitation. Tensor (N.S.) **6** (1956), 69–82.

Some solutions of the field equations of the Einstein generalized relativity theory [The meaning of relativity, 5th ed., Princeton, 1951, Appendix II; MR **17**, 907] are obtained, which can be interpreted as plane electromagnetic waves. Waves of this kind travelling in the same direction can be superimposed, but those in opposite directions cannot. Waves of the present kind can also be superimposed on gravitational waves of the kind discussed earlier by the author [Tensor (N.S.) **6** (1956), 15–25; MR **18**, 704] which however are removable by means of a coordinate transformation. *N. Rosen* (Haifa).

Ernst, Frederick J., Jr. Variational calculations in geon theory. Phys. Rev. (2) **105** (1957), 1662–1664.

Gravitational-electromagnetic entities, called geons, have been studied by J. A. Wheeler using an electronic computor [Phys. Rev. (2) **97** (1955), 511–536; MR **16**, 756]. Idealised spherical geons are now studied by means of an adaptation of the Ritz variational principle. Relevant magnitudes are calculated with the use of simple trial functions, and results are compared with those given by Wheeler. *A. G. Walker* (Liverpool).

Mavridès, Stamatia. Etude algébrique d'un tenseur métrique et d'un champ électromagnétique général en théorie unitaire d'Einstein-Schrödinger. C. R. Acad. Sci. Paris **244** (1957), 1149–1151.

Let $a^{\lambda\nu}$, $b^{\lambda\nu}$, $c^{\lambda\nu}$ be three symmetric (skew symmetric) tensors in a four-dimensional space and let

$$a^{\lambda\nu}=b^{\lambda\nu}+c^{\lambda\nu}.$$

Then the inverse tensor $a_{\lambda\mu}$ may be expressed in terms of the inverse tensors $b_{\lambda\mu}$ and $c_{\lambda\mu}$ and vice versa. The author gives the explicit formulas which may be applied to unified field theory (where there are several possibilities of defining the basic symmetric [skew symmetric] tensor derived from $g_{\lambda\mu}\neq g_{\mu\lambda}$). *V. Hlavatý* (Bloomington, Ind.).

Lenoir, Marcel. Principe d'une théorie unitaire utilisant l'espace fibré des repères affines. C. R. Acad. Sci. Paris **244** (1957), 1151–1153.

On se propose d'étudier les modifications que subit la théorie d'Einstein-Schrödinger lorsqu'on substitue à la connexion linéaire — appelée, en général, connexion affine — une connexion affine définie sur l'espace fibré des repères affines, dont l'origine est caractérisée par un vecteur ξ, aux différents points d'une variété V_4. (Author's summary.) *V. Hlavatý* (Bloomington, Ind.).

Kichenassamy, S. Sur un cas particulier de la solution de $g_{\underset{+-}{\mu\nu};\varrho}=0$. C. R. Acad. Sci. Paris **244** (1957), 2007–2009.

[The results quoted sub 1 are taken from M.-A. Tonnelat "La théorie du champ unifié d'Einstein et quelques-uns de ses développements" Gauthier-Villars, Paris, 1955; MR **17**, 907.] 1) The Einstein connection $\Gamma_\lambda{}^\nu{}_\mu$ is defined by

$$(1)\qquad \partial_\omega g_{\lambda\mu}-\Gamma_\lambda{}^\alpha{}_\omega g_{\alpha\mu}-\Gamma_\omega{}^\alpha{}_\mu g_{\lambda\alpha}=0.$$

Its skew symmetric part $S_{\lambda\mu}{}^\nu$ is given by its covariant components (loc. cit. p. 46)

$$(2)\qquad FS_{\lambda\mu\nu}\ T_{\lambda\mu\nu}.$$

Here F is an explicitly defined scalar and $T_{\lambda\mu\nu}$ is an explicitly defined tensor. $T_{\lambda\mu\nu}$ contains expressions of the type

$$(3)\qquad \varphi\varphi^{\sigma\lambda}\partial_\sigma \log\varphi,\quad \varphi^{\lambda}\partial_\sigma \log\varphi \quad \text{(loc. cit. p. 44)}$$

multiplied by tensors which contain neither φ nor $\varphi^{\lambda\nu}$ ($\varphi^{\lambda\nu}$ is the inverse tensor to $\varphi_{\lambda\mu}=g_{[\lambda\mu]}$ which exists only if $\varphi \overset{\text{def}}{=} \operatorname{Det}((\varphi_{\lambda\gamma}))\neq 0$). The expressions (3) become either meaningless or indeterminate of the type 0/0 for $\varphi=0$ (see the reviewer's remark).

2) The main result of the paper under review is the

claim that (2) is valid also for $\varphi=0$. The author obtains a tensor $T'_{\lambda\mu\nu}$ from $T_{\lambda\mu\nu}$ by a) substituting $\varphi=0$ into T, b) by leaving out the indeterminate terms of the type (3) and (C) by replacing $\varphi^{\frac{1}{2}}\varphi^{\lambda\nu}$ by $\varepsilon^{\lambda\nu\alpha\beta}\varphi_{\alpha\beta}/2$. (see the reviewer's remark). If $F\neq0$, then he claims without proof that $S_{\lambda\mu\nu}=T'_{\lambda\mu\nu}/F$ is the covariant form of the torsion tensor of the connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ for $\varphi=0$. In the remaining part of the paper the author answers a private letter addressed to him by the reviewer.

Reviewer's remark. The above mentioned fallacies stem from the claims a) that $2\varphi^{\frac{1}{2}}\varphi^{\lambda\nu}=\varepsilon^{\lambda\nu\alpha\beta}\varphi_{\alpha\beta}$ is valid also for $\varphi=0$ (because $2\varphi^{\frac{1}{2}}\varphi^{\lambda\nu}$ "is nothing else than the representation of the second term") (loc. cit. 127; cf also pp. 14 and 47), b) that $\varphi^{\frac{1}{2}}\partial_\sigma \log \varphi^{\frac{1}{2}}=\varphi^{\frac{1}{2}}/\varphi^{\frac{1}{2}}\ \partial_\sigma\varphi^{\frac{1}{2}}=0$ for $\varphi=0$ (loc. cit. p. 48). The reviewer failed to understand these peculiar statements (and their applications) which yield easily many amusing contradictions. *V. Hlavatý.*

See also: Gürsey, p. 214; Raman, p. 214; Jauch and Rohrlich, p. 217; Gilbert, p. 228; Bonnor, p. 228; Rindler, p. 228.

Astronomy

Ferrari d'Occhieppo, Konradin. Beitrag zur Bestimmung des Radienverhältnisses bei Bedeckungsveränderlichen. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. II. **165** (1956), 1–39.

A paper by H. Schneller [Veröff. Sternwarte Sonneberg **1** (1949), no. 4, 359–406] contains a method for determining the orbital elements of an eclipsing binary system on the assumption of uniformly bright disks of the two components. The object of this paper is to extend the method to solutions in which allowance is made for darkening at the limb. This requires the calculation of new auxiliary tables which are given in an appendix.
D. Brouwer (New Haven, Conn.).

Contopoulos, G. Der Einfluss des Strahlungsdruckes auf die Dynamik der interstellaren Körner. Z. Astrophys. **42** (1957), 7–33.

A dust-particle of the interstellar medium suffers a repulsion due to the radiation-pressure of a star in its neighbourhood that is much larger than the corresponding effect on a gas atom. This repulsive force, like the gravitational attraction of the star, falls off inversely as the square of the distance from the center of the star. The author works out the path of a dust-particle subjected to these two forces and shows, by means of numerical estimates, that high particle-velocities will be generated, especially by stars of early spectral type. He then considers the motion of a dust-particle through a group of many stars and finds a relaxation time of 4.4×10^8 years and a mean free path of 4500 pc. A resisting force due to the interstellar gas is then introduced into the problem, this resisting force being proportional to the square of the velocity of the dust-particle. The problem is solved by the method of variation of constants. It is shown that the orbits of dust-particles about the center of the galaxy tend to become circular. Considering radial motion from the galactic center, it is shown that the ultimate particle-velocities are very nearly the same as for no resisting-force. Finally, the effect of absorption of radiation is allowed for. The concentration of dust-particles by a single star or by the central part of a galaxy is worked out and its relevance to stellar formation discussed. The radial motion of a single dust-particle is now of the nature of a damped oscillation. *G. C. McVittie.*

Merman, G. A. Zu Arbeiten von R. Vernić über die Regularisierung und die periodischen Lösungen des Dreikörperproblems. Byull. Inst. Teoret. Astr. **6** (1956), 408–415. (Russian. German summary)

Vernić [Diskussion der Sundmanschen Lösung des Dreikörperproblems, Jugoslav. Akad. Znan. Umjet., Zagreb, 1954; MR **16**, 867] claims (loc. cit. p. 86, Theorem 5) to have obtained transformations for the independent variable t which regularize all types of collisions in the general three-body problem. The author shows that Vernić's proof is defective and that, whereas his theorem holds for all binary collisions and for real triple collisions in the case of the Lagrangian motions and in the case of other motions for certain values of the masses, it does not hold for all real triple collisions, and it does not hold for any imaginary triple collision.

Another theorem of Vernić [Hrvatsko Prirod. Društvo. Galsnik Mat. Fiz. Astr. Ser. II. **8** (1953), 247–266; MR **16**, 181] on the non-existence of periodic solutions other than the Lagrangian solutions in the general three-body problem is also shown to be false.

The reviewer would like to point out that in the last differential equation on p. 409 the factor $1/R^3$ is lacking in the first term on the right-hand side. *E. Leimanis.*

Lindblad, Bertil. Differential motions in dispersion orbits in the galaxy. Stockholms Obs. Ann. **19** (1957), no. 9, 15 pp.

In earlier papers [cf. Stockholms Obs. Ann. **18** (1955), no. 6; MR **17**, 419] the author has suggested that nearly circular orbits, in an axisymmetric gravitational potential (V) with a plane of symmetry, occurring in certain zones play a special role in the development of spiral structure. These special regions occur where the variation of the gravitational potential with ϖ (the distance from the axis of symmetry) approximate an inverse square or a quasi-elastic field of force. The reason apparently is that for these fields of force the orbits are closed curves and the author states, "It seems, therefore, that distinct rings of matter are likely to occur preferably in" these zones. Using the observationally deduced variation of the circular velocity in our galaxy, the author determines where these zones occur in an attempt to confirm his theoretical ideas. *S. Chandrasekhar* (Williams Bay, Wis.).

Bellman, Richard; and Kalaba, Robert. On the principle of invariant imbedding and diffuse reflection from cylindrical regions. Proc. Nat. Acad. Sci. U. S. A. 43 (1957), 514–517.

The problem of diffuse reflection and transmission by plane parallel atmospheres has been solved in recent years by the use of certain principles of invariance: e.g. the law of diffuse reflection by a semi-infinite plane parallel atmosphere must be invariant to the addition (or subtraction) of layers of arbitrary thickness to (or from) the atmosphere. [For a general account of these principles, see S. Chandrasekhar, Radiative transfer, Oxford, 1950, chaps. iv and vii; MR **13**, 136.] In the present paper the principle is extended to cylindrical and spherical geometries. In cylindrical geometry, for example, one considers radiation incident on the cylinder in such a way that the direction of incidence on any point is inclined at the same

angle θ to the normal. The reflected radiation should then be a function of the angle φ to the normal. Representing the law of reflection in terms of a scattering function $r(u, v, x)$ where $u=\cos\theta$, $v=\cos\varphi$ and x is the radius of the cylinder, it is clear that one can relate $r(x+dx; u, v)$ to $r(x, u, v)$ by considering the effect of the thin cylindrical shell between x and $x+dx$ which has been added. In this way the following integro-differential equation is derived

$$r_x+\left(\frac{1}{u}+\frac{1}{v}\right)r+\frac{1-v^2}{xv}r_v+\frac{1-u^2}{xu}r_u=$$
$$\frac{\lambda(x)}{4v}+\frac{\lambda(x)}{2v}\int_0^1 r(w, u, x)dw+\frac{\lambda(x)}{2}\int_0^1 r(v, z, x)\frac{dz}{z}$$
$$+\lambda(x)\int_0^1 dwr(w, u, x)\int_0^1\frac{dz}{z}r(v, x, z),$$

where λ is the albedo for single scattering which is allowed to be a function of x; also subscripts denote differentiation with respect to the variable. The foregoing equation must be considered together with the boundary condition $r(v, u, 0)=0$. *S. Chandrasekhar.*

Zeuli, Tino. Equilibrio radiativo di una massa gassosa stellare in lenta rotazione uniforme. Univ. e Politec. Torino. Rend. Sem. Mat. **15** (1955–56), 351–363.

The radiative equilibrium of a uniformly rotating star on the standard model (in which the ratio of the gas pressure to the radiation pressure is a constant and the non-rotating star is a polytrope of index 3) is considered. [The author does not seem to be aware that his paper duplicates an early paper of E. A. Milne, Monthly Nat. Roy. Astr. Soc. **83** (1923), 118–147; see also H. von Zeipel, ibid. **84** (1924), 665–683, 684–701; and S. Chandrasekhar, ibid. **93** (1933), 390–405.] *S. Chandrasekhar.*

Talwar, S. P.; and Tandon, J. N. On the radial pulsation of magnetic stars. Indian J. Phys. **30** (1956), 561–564.

This paper discusses the radial pulsation of a magnetic star. A formula for the period is obtained which is equivalent to the one derived by Chandrasekhar and Limber [Astrophys. J. **119** (1954), 10–13; MR **15**, 750] from the virial theorem. *S. Chandrasekhar.*

Gilbert, C. Dirac's cosmology and the general theory of relativity. Monthly Not. Roy. Astr. Soc. **116** (1956), 684–690 (1957).

The author remarks that the spatial coordinates and time of Newtonian gravitation theory need not be identified with those in which the velocity of light is exactly constant. By assuming different length units to be used in electromagnetic and in gravitational measurements, the ratio of these units varying with time, he derives from an Einstein-de Sitter world-model results similar to those found by Dirac [Proc. Roy. Soc. London. Ser. A. **165** (1938), 199–208].

The author regards as the essentially new concept of his work the statement "The measurements of all quantities occurring in field theory shall be based on units of mass, length and time, derived from the values of the atomic "constants" measured in a local frame with arbitrarily chosen units." *F. A. E. Pirani* (London).

Bonnor, W. B. Non-singular fields in general relativity. J. Math. Mech. **6** (1957), 203–214.

The question of the existence of solutions of Einstein's gravitational equations for empty space, $R_{\mu\nu}=0$. that are devoid of matter and yet are curved, is discussed. It is proved that, in the first approximation, solutions of this kind without singularities and which are flat at spatial and temporal infinity can be found. They are devised by using Synge's method of solving the wave-equation. An exact solution with cylindrical symmetry is then obtained which is interpreted as a gravitational wave that comes in from an infinite distance from the axis of symmetry in the infinitely remote past. The wave grows in intensity until it passes through the axis, and then disperses again to spatial infinity at the end of an infinite time. Space-time has no singularities anywhere, is flat at an infinite distance from the axis and for infinitely large values of the time. But the behavior of the solution cannot be determined for infinite values of the coordinate parallel to the axis. Thus the solution may involve matter in regions defined by infinite values of the axial coordinate. *G. C. McVittie* (Urbana, Ill.).

Rindler, W. Visual horizons in world-models. Monthly Not. Roy. Astr. Soc. **116** (1956), 662–677 (1957).

In a model of the universe with metric

$$ds^2=c^2dt^2-R^2(t)\,\frac{dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2}{(1+kr^2/4)^2}$$

two kinds of "horizons" are defined: (a) the event-horizon which is a hypersurface in space-time on one side of which lie all events that have been, are being or will be observed by a given fundamental observer, and on the other lie the events which are forever outside the observer's possible cognisance; (b) the particle-horizon, which is a corresponding hyper-surface separating those fundamental particles that the observer has already observed, or is observing, at a specific instant t_0 from those that he cannot yet observe. Fundamental observers and particles have world-lines characterised by constant coordinates (r, θ, ϕ). For an event-horizon, $\int^\infty dt/R(t)$ must converge to a finite limit; for a particle-horizon, $\int_0 dt/R(t)$ or $\int_{-\infty} dt/R(t)$ must converge, according as $t=0$ or $t=-\infty$ is the initial instant in the model universe. The consequences of these definitions are traced out: if an event-horizon exists, then particles that have at some time been visible to the observer, remain so forever; particles at some instant within the event-horizon must eventually pass beyond it at the local speed of light. If a particle-horizon exists more and more particles become visible to the observer as time goes on; the horizon is the boundary of the observer's light cone at the initial instant.

Numerous specific examples are worked out for illustrative purposes including one in which both types of horizon exist. When this happens in a model, an absolute horizon exists beyond which lie events unobservable to a given observer, even if he journeys in such a way that his world-line is not a fundamental one. The last two sections deal with the question of horizons in kinematical relativity. *G. C. McVittie* (Urbana, Ill.).

Berlage, H. P. The basic scheme of any planetary or satellite system. Nederl. Akad. Wetensch. Proc. Ser. B. **60** (1957), 75–87.

Expressions for the density at any point of a gaseous disk revolving around a central body of preponderant mass were derived by the author in earlier papers [same Proc. **35** (1932), 553–562, **38** (1935), 857–863]. In addition to the central mass M, two constants are introduced, the central density ρ_0 and a constant a which defines the exponential decrease of density ρ_e in the equatorial plane

with increasing distance from the axis of rotation. By making the assumption that the product of the gas constant R and the absolute temperature T is constant everywhere in the disk, integrations can be performed to give the total mass m, the total angular momentum θ, the total kinetic energy U and the total potential energy V of the disk as functions of ρ_0 and a. The density of the equatorial plane,

$$\rho_e=\rho_0 \exp[-ar^{\frac{1}{2}}],$$

is then modified to

$$\rho_e+\Delta\rho_e=\rho_0(1+\varphi)\exp\left[-(a+\psi)r^{\frac{1}{2}}+\varepsilon \cos\left(p \ln\frac{r}{r_m}\right)\right].$$

With φ, ψ/a and ε small quantities, this represents a small undulation superimposed on the original density distribution. The equations resulting from the conditions

$$\Delta m=0,\ \Delta\theta=0,\ \Delta(U+V)=0,$$

after elimination of φ and ψ, yield an equation from which p is solved as a function of r_m. The resulting density maxima provide schemes for planetary and satellite systems. Comparisons between observed and computed distances are discussed in the second half of the paper.

D. Brouwer (New Haven, Conn.).

See also: Kulikov, p. 183; Bohan, p. 183.

Geophysics

Morgan, G. W. On the wind-driven ocean circulation. Tellus 8 (1956), 301–320.

Theories of the wind-driven ocean in closed ocean basins are generally based upon the assumption that the principal factors involved are the wind, Coriolis parameter, lateral friction, and pressure forces. The resulting linear problem gives a solution which, in order that the correct width of the current on the western shore may be obtained, involves a larger value of lateral eddy viscosity than normally accepted.

In this paper the author considers first the necessity for viscosity and then proposes an ocean model in which a closed ocean is divided into three regions. These are a northern band in which nonlinear viscous and nonsteady effects are included, a western frictionless stream region including nonlinear inertia, and lastly, a large "interior" region where nonlinear terms and viscosity may be neglected. The equations of motion are derived in the two cases of a uniform motion and one composed of two superimposed layers each of uniform density. It is shown that both models applied to the North Atlantic give a reasonable width for the Gulf Stream and that, in the interpretations of the results, much depends upon the variation of the Coriolis parameter. *D. C. Gilles.*

Rouaud, A. Application d'une correction de sphéricité de la Terre dans la détermination du vecteur vent. J. Sci. Météorol. 8 (1956), 117–132 (1957). (English and Spanish summaries)

La direction et la vitesse du vent en altitude se déterminent à partir de la projection sur la sphère terrestre de la trajectoire d'un ballon libre, lâché dans l'atmosphère.

En exploitation courante, la projection s'effectue, pour des raisons de commodité et de rapidité, non sur une sphère, mais sur un plan à une échelle donnée. Une telle manière de procéder entraîne une erreur systématique qui n'est plus négligeable lorsque le ballon s'éloigne beaucoup, ce qui se produit assez fréquemment maintenant que les méthodes radioélectriques permettent de suivre le ballon à haute altitude. Si en particulier le ballon est suivi au radar, le fait de négliger la sphéricité de la Terre entraîne une erreur d'altitude qui devient sensible lorsque la distance horizontale du ballon dépasse 25 kilomètres.

L'auteur cherche à établir une méthode graphique simple pour déterminer la correction, due à la courbure terrestre, qu'il faut appliquer lorsqu'on effectue une projection orthogonale sur un plan. La solution a été trouvée en prenant pour paramètre la distance horizontale non corrigée que l'on obtient très simplement soit à l'aide de la projection, soit par le calcul. On a utilisé des abaques logarithmiques pour n'avoir à tracer que des droites, ce qui facilite l'interpolation.

L'auteur montre qu'il n'y a pas lieu de tenir compte de l'erreur de direction du vecteur vent, due à la courbure terrestre; cette erreur est dans tous les cas négligeable devant les erreurs dues aux autres causes.

M. Kiveliovitch (Paris).

Malkevič, M. S. Method of theoretical determination of the vertical temperature gradient of the air. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1956,** 676–688. (Russian)

L'étude théorique de ce problème se ramène à la recherche d'une solution des équations intégrales dont le noyau représente une combinaison des fonctions transcendantes de Gold. L'auteur se propose de remplacer ces méthodes trop compliquées par une autre plus simple et plus maniable par approximation en remplaçant le noyau par des fonctions exponentielles et ensuite l'équation intégrale par une équation différentielle équivalente.

M. Kiveliovitch (Paris).

Gates, John P. Descriptive geometry and the offset seismic profile. Geophysics 22 (1957), 589–609.

An illusion is created in attempting to portray steeply dipping seismic data by standardized geometric procedures. The offset seismic profile, although resembling the vertical geologic cross-section in appearance, may have an inherent tilt which is a function of cross-dip. Unless this tilt is accounted for during the solution of critical structural problems, serious errors can enter into the interpretation of the seismic data. Descriptive geometry procedures are applied to establish the tilt and determine its effects on fault traces, structural axes, and well ties.

Author's summary.

Kazinskiĭ, V. A. On the calculation of the effect of topographical masses on subterraneous gravitational measurements. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1957,** 30–38. (Russian)

The author expounds a method for the calculation of topographical mass distributions which affect gravitational measurements. Finite difference methods are employed. Some of the results are given in tabular form.

It should be noted, however, that the proposed method lacks mathematical rigour. Convergence, and other essential criteria are not touched upon.

K. Bhagwandin (Oslo).

Ansermet, A. Sur de nouvelles méthodes de calcul en topographie. Bull. Tech. Suisse Romande 83 (1957), 50–54.

This paper is an expository account of changes that

have evolved over the past few years in the computation and adjustment of topographical or geodetic networks. The changes the author is chiefly concerned with are the substitutions in many cases of linear measures for angular measurements, change of variables in calculation of the adjustments and the use of graphical mediums in the calculations. Formation of residual equations with suggested methods of solution and examples are given. Reference is made to the work of C. F. Baeschlin [Die Berechnung von Streckennetzen, 1951], A. Frank [Z. Vermessungswesen **69** (1940), 97–112, 145–160, 193–204] W. Grossmann [Grundzüge der Ausgleichungsrechnung···, Springer, Berlin, 1953; MR **15**, 650], W. Jordan [Handbuch der Vermessungskunde, Bd. I, 4. Aufl., Metzler, Stuttgart, 1894], and A. Ansermet [Rev. Suisse Mensurations **1956**, no. 2, 8]. *P. D. Thomas* (Washington, D.C.).

See also: Zmuda, p. 174; Penndorf and Goldberg, p. 181; Walters, p. 203; Scheidegger, p. 208; Goody, p. 213.

OTHER APPLICATIONS

Economics, Management Science

van Dantzig, D. Economic decision problems for flood prevention. Econometrica **24** (1956), 276–287.

The paper summarizes a part of the contribution made by the Mathematical Center at Amsterdam to the work of the "Delta Commission" of hydraulic engineers appointed by the Netherlands government to design measures for preventing the repetition of the flood disaster of 1953. The optimal amount $X=\hat{X}$ by which Holland's dikes should be heightened (disregarding local variations) minimizes $I(X)+L(X)$, where $I(X)$ is the cost of heightening the dikes and subsequently adding to the height periodically, to counteract the secular sinking of land which is approximately $kX+\text{const}$, and $L(X)$ is the expected and discounted value of loss in case of disaster,

$$L(X)=\frac{100V}{\delta'-\alpha\eta}\cdot p_0\exp(-\alpha X)\cdot\frac{1-\exp(\alpha\eta-\delta')T}{1-\exp(-\delta'T)};$$

here V is the loss incurred whenever sea level exceeds the height of dikes, $p_0\exp(-\alpha X)$ the empirically estimated probability that sea level exceeds X (with parameters p_0 and α subject to possible adjustments for the higher values of the argument), η the rate of secular sinking of land, T the period for re-heightening of dikes, and δ' the difference between interest rate and the rate of growth of national wealth. $\hat{X}$ is obtained by differentiation. Since the numerical constants are very uncertain, their most unfavorable estimates were used but the resulting "pessimistic" outcome subsequently adjusted.

Some unresolved basic problems are also discussed: It is possibly inadequate to extremize expected monetary value rather than expected utility; or indeed, to use the expected value concept itself where the number of comparable events is small. It is difficult to estimate interest rates, and to compare national wealth, over centuries, and to put values on lost human lives. The quantity V is not a linear function of the area flooded (as has apparently been assumed) but is rather S-shaped; etc. The original report is Math. Centrum Amsterdam. Statist. Afdeling Rep. **1953-32(3)**. *J. Marschak.*

Theil, H. Linear aggregation in input-output analysis. Econometrica **25** (1957), 111–122.

Suppose that the technical structure of firms is exactly (up to a random error) determined by constant input-output coefficients and that the firms are grouped into industries. The author studies the bias in prediction of outputs from final demands due to the aggregation procedure, i.e., the macro-predictions are made by assuming that the input-output table computed from the industry totals of inputs and outputs is a constant. Let (μi) denote the ith firm in the μth industry ($i=1,\cdots,I_\mu$; $\mu=1,\cdots,M$), ω the total number of firms, $A_{\omega\omega}$ the matrix of input-output micro-coefficients (for flows among firms). Then if the aggregated matrix is used to predict industry outputs from a vector x_M of final demands by industries, the vector e_M of errors in prediction (neglecting random errors) is given by

$$e_M=I_{M\omega}[(I-A_{\omega\omega}Z)^{-1}-(I-A_{\omega\omega})^{-1}]X_\omega,$$

where $I_{M\omega}$ is a matrix of M rows and ω columns, with 1 in row μ and column $\mu'i$ if $\mu=\mu'$ and 0 otherwise, Z is an $\omega\times\omega$ matrix with blocks Z_μ down the diagonal and 0's elsewhere, Z_μ is the square matrix of order I_μ with all the elements of the ith row being $\zeta_{\mu i}$, the fraction of industry μ's output contributed by firm (μi), and x_ω is the vector of final demands for the outputs of individual firms.

Some properties of the matrix which multiplies x_ω are discussed and economic interpretations given. A number of special cases are analyzed. *K. J. Arrow.*

Greenhut, Melvin L. A general theory of plant location. Metroecon. **7** (1955), 59–72.

Stress is laid on the necessity of considering both cost and demand factors simultaneously in a theory of spatial location. *K. J. Arrow* (Stanford, Calif.).

Pfouts, R. W. Some difficulties in a certain concept of community indifference. Metroecon. **7** (1955), 16–26.

Suppose that there is a change of economic state such that some individuals gain (in utility terms) while others lose. The question arises as to a system of redistribution of the goods in the new state such that each individual is indifferent to his position in the original state. The author shows that in general such redistributions are not uniquely determined. *K. J. Arrow* (Stanford, Calif.).

Tedeschi, Bruno. Sulla teoria dei capitali accumulati. Giorn. Ist. Ital. Attuari **19** (1956), 92–130.

A historical survey of various functional equations satisfied by life insurance functions, e.g. Thiele's differential equation and integral equation of the Volterra type. *P. Johansen* (Copenhagen).

Stevens, W. L. Bias of the census method. J. Inst. Actuar. Students' Soc. **14** (1957), 192–198.

Two conditions have been proposed by Anderson and Dow for the census method of constructing a mortality table to be strictly correct: that at the beginning and end of the investigation, ages of lives are evenly spread over the year of age x to $x+1$, or that numbers and distributions ages are the same at beginning and end.

The author proves that the second condition gives a strictly unbiased estimate of the central rate of mortality and if the first condition is satisfied, the estimate is biased with an error of little practical consequence. *W. Saxer* (Zurich).

Bishop, George T. On a problem of production scheduling. Operations Res. 5 (1957), 97–103.

The author shows how a method recently developed by Prager, Management Science, 13 (1956), pp. 15–23, can be used to simplify the computational solution of a class of problems arising in production scheduling. *R. Bellman.*

★ **Marschak, Thomas. Centralization and decentralization in economic organizations.** Technical report no. 42, prepared under contract N6onr-25133 for Office of Naval Research. Department of economics, Stanford University, Stanford, Calif. 1957. 252 pp.

The author seeks to formalize the concepts of centralization and decentralization in economic organizations, which might be national economies or individual firms. The various elements of a decision system are given scrupulous definition: the decision variables, the messages among the participants, the rules which govern the computations made by each participant and which lead to subsequent rounds of decision variables and of messages. Decentralization and centralization are defined in terms of the rules for sending information from one participant to another and the roles of the different participants in choosing the decision variables.

There is a discussion of the criteria for choice among decision systems. These depend upon the speed with which an optimum is approached, as defined by the loss of utility due to making decisions before the optimum is reached. Other considerations are the limitations upon computational abilities of the participants and the number of channels required for the transmission of messages. Since the system has to be prepared to meet many different sets of exogenous conditions, the choice has to be based on some criterion for choice under uncertainty, such as a Bayesian or a minimax rule. The various possibilities are explored, and the difficulties of finding a suitable simple criterion exhibited.

The general concepts are applied to two (artificially defined) organizations. In each one a number of alternative decision systems are considered and their operating characteristics found. In each case it was found that hybrid system which differed from both the pure centralized and pure decentralized system were frequently advantageous in economizing upon computational and message facilities and in increasing the speed of convergence.

K. J. Arrow (Stanford, Calif.).

Hale, Jack, K.; and Reed, Ronald L. A formulation of the decision problem for a class of systems. Naval Res. Logist. Quart. 3 (1956), 259–277 (1957).

The authors investigate the problem of deciding which of a given set of systems is best by formulating the problem precisely and obtaining a solution which is essentially independent of the criteria used. For example: to compare two weapons use x strikes at a target of the first and y strikes of the second. Let $V(x, y)$ be the value-of-destruction function and let $C(x, y)$ be the weapons cost function. Let D be an arbitrary criterion and assume that x and y are continuous variables. If all the partial derivatives of V and C with respect to x and y are positive, the behavior of D along $V(x, y)=V_0$ or $C(x, y)=C_0$, for every D with partial derivatives satisfying $D_V>0$, $D_C<0$ is determined by the sign of the function $V_x/C_x-V_y/C_y$. If the latter is positive the larger value of D will occur in the direction of increasing x and if it is negative the larger value of D will occur in the direction of increasing y. Thus this function can be used to determine the points (x, y) on which D is maximum. *T. L. Saaty* (Washington, D.C.).

See also: Spring and Leepin, p. 183; Yavlinskiĭ, p. 183.

Programming, Resource Allocation, Games

Savage, I. Richard. Cycling. Naval Res. Logist. Quart. 3 (1956), 163–175 (1957).

The author considers three problems of scheduling a sequence of times at which certain routine operations should be performed in order to minimize expected losses. In all cases certain cost functions or characteristics of these functions are assumed known. The following problems are considered. (1) Replacement: Minimize long-run average-cost-per-unit time of replacing certain parts at one time. (2) Inventory: Minimize yearly costs from taking one or more inventories per year plus imperfect information between them. (3) Preparedness: Minimize expected costs required to prepare stored equipment for emergency use. All of these problems involve a fixed cost A (replacement, taking inventories, or inspecting equipment) plus the total loss, $F(x)$, due to scheduling, the latter being a function of the scheduling program, e.g. the time (x) between replacements or taking inventories. For differentiable $F(x)$, the following equation (or extension of it) must be solved

$$xF'(x)=A+F(x).$$

R. L. Anderson (Raleigh, N.C.).

★ **Vajda, S. The theory of games and linear programming.** Methuen and Co. Ltd., London; John Wiley and Sons, Inc., New York, 1956. v+106 pp. $1.75.

This is a short expository book on Linear Programming and Game Theory with a strong emphasis on computational methods. It is aimed at the non-expert and assumes no prior knowledge of either topic in the title. The mathematical prerequisites are "some knowledge of fairly elementary algebra and of linear analytic geometry".

The title of the book is rather misleading since the only games whose theory is discussed are finite two-person zero-sum games in normal form ("matrix "games) which comprise but one branch, albeit the main one, of the subject. The same criticism applies to the first chapter, entitled "An Outline of the Theory of Games," in which the basic definitions for matrix games are illustrated by means of a number of illuminating numerical examples. The second chapter, "Graphical Representation," is devoted to giving analytic-geometric pictures for matrix games, including an intuitive proof of the fundamental theorem for the case of a 2 by 3 game. Chapter III contains the proof of the fundamental theorem as given in von Neumann and Morgenstern, "Theory of games and economic behavior" [2nd ed., Princeton, 1947; MR 9, 50]. Chapter IV, "An Outline of Linear Programming", defines the problem, illustrates it by a number of examples, and proceeds immediately to solve a numerical example by the simplex method. There follows a chapter on graphical representation of linear programming, followed by two chapters on the simplex method, including discussion of degeneracy and possible nonexistence of feasible or optimal solutions. Chapter VIII, called "Duality," contains what the author asserts to be a proof of the Fundamental Duality Theorem (the capitals are his). He then describes the game corresponding to a linear program and concludes with a discussion of the Dual Simplex Method of Lemke. Chapter IX, "The

Solution of Games," illustrates by a numerical example how to use the simplex method in solving a matrix game. There follows a second chapter on graphical representation, and the final chapter gives an outline of a second algorithm for solving programs, the Method of Leading Variables of Beale.

The reviewer found the exposition on computational methods quite difficult to follow, one reason being the tangle of subscripts in expressions such as

$$\sum_{k=1}^{n-m} z_{u_s u_{m+k}} x_{u_{m+k}},$$

necessitated by the author's apparent desire to avoid at all costs the use of vectors. The interested reader might compare this treatment of the simplex method, for instance, with the comparatively uncluttered typography used by Dantzig in his original paper [Activity analysis of production and allocation, Wiley, New York, 1951, pp. 339–347; MR 15, 47].

On the whole, the book's point of view is that of the engineer interested chiefly in numerical answers, rather than of the mathematician whose primary concern is with the theory. As evidence of this, the only place in which formal proofs are given are in the sections taken from von Neumann and Morgenstern [loc. cit.]. Nevertheless, the first five chapters provide a good general introduction to the basic ideas in the subject. The reader who wishes to go further into these matters, however, is better advised to consult original sources. *D. Gale.*

Vajda, S. An outline of linear programming. (Symposium on linear programming). J. Roy. Statist. Soc. Ser. B. 17 (1955), 165–172; discussion, 194–203.

In this expository paper, linear programming and examples of its applications are given. One family of computational methods (the Simplex Method and its relations) is discussed. *H. W. Kuhn* (Bryn Mawr, Pa.).

Karush, W.; and Vazsonyi, A. Mathematical programming and service scheduling. Management Sci. 3 (1957), 140–148.

For time intervals $i=1, \cdots, N$ we are given service level requirements r_i (also given for $i=0, N+1$), increasing convex polygonal production cost functions $f^{(i)}$, and a change of production cost function $h(v)=av^+ + bv^-$. The problem is to choose $x_0, \cdots, x_{N+1}$ to minimize $\sum_1^N f^{(i)}(x_i)+\sum_1^{N+1} h(x_i-x_{i-1})$ subject to $x_0=r_0$, $x_{N+1}=r_{N+1}$, and $x_i \geq r_i (1\leq i\leq N)$. The authors give an algorithm for constructing an optimum solution. Generalizations are mentioned. *J. Kiefer.*

Dantzig, George B. Recent advances in linear programming. Management Sci. 2 (1956), 131–144.

★ **Dantzig, George B. Developments in linear programming.** Proceedings of the Second Symposium in Linear Programming, Washington, D.C., 1955, pp. 667–685. National Bureau of Standards, Washington, D.C., 1955.

Both of these papers cover essentially the same ground, the second being an expanded version of the first. The material presented covers the status of linear programming research at the beginning of 1955, a topic which no one is better qualified to expound than the father of the subject. The particular items mentioned include: variables with upper bounds minimization of convex functions, linear programming under unertcainty, applications of linear programming to combinatorial theorems and algorithms, special methods for special problems, and special methods for dynamic and block triangular problems. All the points covered have appeared in the literature either before publication of this paper or subsequently, and there has been considerable subsequent progress on some of the topics which had not already reached their ultimate development, justifying the author's optimism about the steady growth of the field. *A. J. Hoffman.*

★ **Heselden, G. P. M.; and Vajda, S. Linear programming of an air-lift. With an appendix by E. M. L. Beale.** Conference on linear programming, May, 1954, pp. 5–18; discussion, 19. Ferranti Ltd., London.

Most of this paper is devoted to an exposition of the simplex method applied to a toy airlift problem. In addition to solving the problem as stated, the authors point out how to modify the problem, and how the answers change, if it is desired to avoid large fluctuations between the amount delivered in successive weeks. In an appendix E. M. L. Beale considers the following problem: Let L_1 and L_2 be linear forms defined on a convex set. Under what circumstances is it true that a point in the convex set which minimizes L_1 for a fixed value of L_2 also maximizes L_2 for a fixed value of L_1? The problem is solved by considering the planar graph of all the possible values of (L_1, L_2) attained on the convex set.
A. J. Hoffman (New York, N.Y.).

Zubrzycki, S. On the game of Banach and Mazur. Colloq. Math. 4 (1957), 227–229.

In the Banach-Mazur game for a set A of positive numbers, the two players alternately choose the terms of a series of strictly decreasing positive numbers; the first player wins if and only if the series is convergent with sum an element of A. The author shows that the second player wins if A is countable. *J. Isbell* (Seattle, Wash.).

Zięba, A. An example of the game of Banach and Mazur. Colloq. Math. 4 (1957), 230–231.

Notation as in preceding review. If A consists of every other interval $[1-2^{-n}, 1-2^{-n-1})$ the second player wins, but if the single point 1 is added, the first player wins.
J. Isbell (Seattle, Wash.).

Brown, Richard H. The solution of a certain two-person zero-sum game. Operations Res. 5 (1957), 63–67.

For $0<b$ and $0<\theta<1$, (x, y) is chosen according to $F\times G$ on the rectangle $[0, b+1]\times[0, b]$ with expected payoff

$$M(F, G)=(1-\theta)F\times G(x\leq y)+F\times G(y<x\leq y+1).$$

With $n=[b]$, the value $v=(1+\theta+\cdots+\theta^n)^{-1}$ and a pair of good strategies are implicit in the relationship of the telescoping sums,

$$\sum_{y=0}^{n}\theta^{n-y}M(F, y)\leq 1=\sum_{x=1}^{n+1}\theta^{x-1}M(x, G).$$

{It is noteworthy that all of I's good strategies are dominated; $M(F, b)\geq v$ if and only if $F(x>b)\geq\theta^n v$.}
J. Hannan (East Lansing, Mich.).

Mycielski, Jan; Świerczkowski, S.; and Zięba, A. On infinite positional games. Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 485–488.

In this note the authors announce several results on

various infinite games with perfect information. The first games considered are general infinite move games of the type introduced by Gale and Stewart [Contributions to the theory of games, v. 2, Princton, 1953, pp. 245–266; MR 14, 999]. Among the results announced are: if the winning set S is an F_σ or G_δ, then the game Γ is determined [this was previously proved by Wolfe, Pacific J. Math. 5 (1955), 841–847; MR 17, 506]; if the space of plays is compact and the pay-off function continuous, then Γ is determined (previously proved by Gale and Stewart, ibid.); if the pay-off function is of first Baire class, then Γ is determined.

Next they consider the game of Mazur, in which A and B form a partition of [0, 1]. The players alternately choose non-degenerate closed intervals, each being a subinterval of the interval previously chosen. Player II wins or loses according as the intersection of the intervals chosen lies in B or not. Theorem: This game is a win for I if and only if B is of first category at some point. It is a win for II if and only if A is of first category.

The final section is devoted to continuous move games, motivated by games of pursuit. A general strict determinary theorem is announced. *D. Gale.*

See also: Fort, p. 152; Saline, p. 175; Morton and Land, p. 175; Mathematical models of human behavior, p. 233.

Biology and Sociology

Aitchison, J.; and Silvey, S. D. The generalization of probit analysis to the case of multiple responses. Biometrika 44 (1957), 131–140.

If a concentration x of a poison is administered to an individual, it can produce one of two results, death or survival. Probit analysis was developed to provide a method of analysing this phenomenon statistically, using the convenient and reasonable assumption that the probability of death is of the form Φ $(a \log x+b)$, where Φ is the normal integral, and a, b are constants to be estimated. More generally it will cover any two-fold classification in which the probability of falling into one class is $\Phi(f(x))$, where $f(x)$ is any sufficiently simple function of the 'intensity of treatment' x which is controllable at will. Here a generalization is proposed in which an increase in x causes an individual to pass successively through a sequence of more than 2 distinct classes: under suitable assumptions, the analysis is strictly analogous to probit analysis. A numerical example is given of some data on the passage of an insect through various stages, the variable x being the age of the insect.
Cedric A. B. Smith (London).

Bartlett, M. S. On theoretical models for competitive and predatory biological systems. Biometrika 44 (1957), 27–42.

The author discusses a number of deterministic and stochastic models for the development of mixed populations of two species. The main points on which attention is focussed are the behaviour of small oscillations in the deterministic models, and the probability of estimation of one or other species in the stochastic models. After a preliminary consideration of a logistic model for population growth of one species, the situations considered are (a) the classical prey-predator system, with consideration of the effects of a lag in births (which tends to increase the probabilities of extinction) and of immigration (which reduces the probabilities); and (b) the competition between two species. Illustrations are taken from biological data and from Monte Carlo realizations. *P. Armitage.*

Danziger, Lewis; and Elmergreen, George L. Mathematical models of endocrine systems. Bull. Math. Biophys. 19 (1957), 9–18.

Die Verfasser arbeiten eine verallgemeinerte mathematische Theorie der Funktion des endokrinen Drüssensystems aus. Das Hormon einer jeden Drüse steigert oder hemmt die Sekretion von anderen innersekretorischen Drüsen, (doch muss wenigstens die Sekretion von einem Hormon unabhängig von den übrigen sein). Der auftretende Prozess von ineinandergreifenden Reaktionen kann dann ganz allgemein durch das folgende System von Differentialgleichungen erster Ordnung beschrieben werden:

$$\frac{dx_i}{dt}+\lambda_i x_i=Q_i \quad (i=1, 2, 3, \cdots, n),$$

wo die Konstante λ_i das Verschwinden der Komponente (des Hormons) x_i charakterisiert und Q_i bedeutet die Beeinflussung der Produktion dieses Hormons durch die anderen Hormone, also ist

$$Q_i=A_{i0}+\sum_{j=1}^{n} A_{ij}x_j.$$

A_{i0} bedeutet die unabhängige Produktion des fraglichen Hormons und kann gleich Null oder positiv sein. Eine weitere Bedingung ist, dass Q_i auch positiv und auch negativ sein kann, wenn x_i ein Enzym ist, ist es jedoch ein Hormon, so muss $Q_i \geqq 0$ sein. Ausserdem haben nur Werte $x_i \geqq 0$ einen Sinn.

Weiter wird dann gezeigt, dass ein System von zwei Drüsen in einer Art Push-Pull Schaltung, wie in der Radiotechnik, das durch das Gleichungsystem

$$\frac{dx_1}{dt}+\lambda_1x_1=a_{10}-a_{12}x_2=Q_1 \quad (Q_1 \geqq 0),$$
$$\frac{dx_2}{dt}+\lambda_2x_2=a_{21}x_1$$

beschrieben wird, zu keinen Relaxationsschwingungen fähig sein kann. In einem analogen System von drei (oder vier) Differentialgleichungen kann das jedoch der Fall sein. Zur Untersuchung dieser Frage werden die Stabilitätskriterien von Routh herangezogen. Dieses Problem ist im Zusammenhang mit der Deutung der Schizophrenie (Katatonie) und des normalen Menstruationszyklus von Bedeutung. *T. Neugebauer.*

★**Mathematical models of human behavior.** Proceedings of a symposium. Sponsored by Dunlap and Associates, Inc. and The Commission on Accidental Trauma, Armed Forces Epidemiological Board, 1955. vii+103 pp.

This volume contains a series of papers presented at a Symposium devoted to the development of methodological techniques for studying human behavior with a hope for eventual application to the study of accident processes. As might be expected from the title, a wide range of mathematical models and techniques are discussed.

Let us briefly review the contents: 1. (M. M. Flood) This paper discusses the design of experiments constructed to test various questions involved in social organization. The treatment is descriptive with a tabulation of the results of some experiments, and there is no theoretical discussion. 2. (R. R. Bush) This is a brief description of

some mathematical models of learning processes following the work of Bush and Mosteller [Stochastic models for learning, Wiley, New York, 1955; MR 16, 1136] Burke and Estes, and Miller and McGill. 3. (H. H. Jacobs) In this paper the author discusses some stochastic models of accident processes which can be constructed along the line of a linear contagion scheme. The availability of fundamental parameters from person to person makes it difficult to construct a widely applicable theory. 4. (R. D. Luce) This paper presents a brief discussion of two approaches to the theory of n-person game theories, the Von Neumann-Morgenstern theory and the theory of Shapley, together with some of the difficulties associated with these approaches. He then introduces a promising method based upon dynamic versions. There are, however, a number of difficulties still to be resolved. 5. (J. Marschak) The author considers the problem of introducing a metric into a set of incommensurable or partially incommensurable quantities, a problem which occurs in the social sciences with annoying regularity. He introduces a mathematical technique for treating these questions. 6. (H. Markowitz) The author shows that a number of phenomena observed in the study of lotteries, insurance investigations and gambling can be explained by means of utility curves with three points of inflection, together with additional hypothesis. 7. (W. E. Estes) This is a short paper on the structure of mathematical models of learning processes with a discussion of the ideas behind particular models used by the author and his collaborators [cf. "Generation of new experimental phenomena by a theory of stimulus variability in learning", Symposium paper, Amer. Psychol. Assoc., 1953] 8. (C. H. Coombs and R. C. Kao) This paper is a further discussion of a central problem of evaluating situations involving partially or totally incommensurable quantities. Here the emphasis is on the over-all effect of different motivations. 9. (M. E. Jarvik) The author discusses the problem of explaining observed gambling habits in terms of psychological versus objective probability. The discussion here has intimate connection with the concepts introduced by Markowitz in his treatment of utility curves. In this paper the emphasis is more upon the psychological and philosophical aspects of the problem. 10. (W. Edwards) This paper lies in the same general domain as the paper above and the paper of Markowitz referred to previously. Here the point under discussion is that of determining experiments to obtain the utility curves. 11. (P. Lazarsfeld) In these concluding remarks the author covers two important and controversial points. The first is that of the value of mathematical models in the social and unsocial sciences, and the second is the problem of the proper domain of the social experimenter. Concerning the second, he makes the remark that the proper study of the psychology of man may not be the psychology of rats, and equivalently the proper way to study the behavior of large groups may not be to study small groups.

R. Bellman (Santa Monica, Calif.).

Smith, Nicholas M., Jr. A calculus for ethics; a theory of the structure of value. I, II. Behavioral Sci. 1 (1956), 111–142, 186–211.

The first part, using a stochastic model, concerns itself with the most general application of values such as their occurrence in decision procedures, supplying a number of illustrative examples. The second part discusses ethical systems, e.g., Utilitarianism, Casuistry, Moral idealism, Hegelianism and Pragmatism, in terms of the theory of the first part. The principle of analogical conformity, asserting that all rational forms including many natural social-science, and ethical laws are homomorphic to a general form, which is a generalization of probability theory, is proposed as a unifying rational principle.

T. L. Saaty (Washington, D.C.).

Estes, W. K. Theory of learning with constant, variable, or contingent probabilities of reinforcement. Psychometrika 22 (1957), 113–132.

A single stimulus is presented in repeated trials and each response is followed by a "reinforcing event". To each of the possible responses A_j there corresponds a distinguished reinforcing event E_j. The E_j's modify the probabilities $p_{i,n}$ of A_i occurring on the nth trial so that

$$p_{i,n+1}=(1-\theta)p_{j,n}+\theta\delta_{jk} \quad (0<\theta\leq 1)$$

if E_k follows A_j on the nth trial.

The probability $\pi_{ij,n}$ that E_j will occur if A_i does on the nth trial may or may not depend on i or n. Asymptotic and transient behavior of the model is studied in a number of cases including $\pi_{j,n}$ constant, linear in n (in an appropriate range), $\pi_{j,n}=a_j+c_jb_j^n$, and $\pi_{j,n}$ periodic in n. The contingent case in which the occurrence of E_j on trial n depends on the outcome of trial $n-r$ is also discussed with a 2-person game situation in mind. The model is proposed as a descriptive device and may be useful in describing adaptation under changing reinforcement situations.

M. L. Minsky (Cambridge, Mass.).

Harris, William P. A revised law of comparative judgment. Psychometrika 22 (1957), 189–198.

Computationally the problem is the elementary one of obtaining a least squares solution A_i and B_j of

$$A_i-B_j=X_{ij},$$

where $i\leq n$, $j\leq m$ but some equations may be missing. The proposed procedure amounts to a single-step iteration. The discussion is mainly devoted to psychological applications.

A. S. Householder (Oak Ridge, Tenn.).

Rodgers, David A. A fast approximate algebraic factor rotation method to maximize agreement between loadings and predetermined weights. Psychometrika 22 (1957), 199–205.

Mathematically the author discovers that the scalar product of a unit vector with a given vector is maximal when the unit vector has the direction of the given vector.

A. S. Householder (Oak Ridge, Tenn.).

See also: van Elteren and van Peype, p. 187; Kestin and Runnenburg, p. 187.

Information and Communication Theory

Crick, F. H. C.; Griffith, J. S.; and Orgel, L. E. Codes without commas. Proc. Nat. Acad. Sci. U. S. A. 43 (1957), 416–421.

A set of ordered triads, each consisting of 3 of the letters A, B, C, D (repetitions being allowed), is to be selected to satisfy the following condition. Any sequence of letters which is formed by placing a sequence of triads in linear order, must be interpretable as such in a unique way; i.e. any group of 3 adjacent letters which are members of two rather than one triad must not itself be a member of the

original set of triads. It is shown that the maximum number of triads in the set is 20, and some solutions when this number is 20 are given. The physical interest in this problem lies in theory that amino acids (triads) are ordered on a nucleic acid strand, and that the order of the amino acids is determined by the order of the nucleotides (letters) of the strand. There are 4 different nucleotides and about 20 different amino acids.

P. Armitage (Bethesda, Md.).

Cohen, J. W. Certain delay problems for a full availability trunk group loaded by two traffic sources. Communication News **16** (1956), 105–113.

A group of N trunks handles traffic from two sources described by independent Poisson processes. It is said that the trunk holding-times have a Poisson distribution, but from the (correct) results of the paper it is clear that the author meant negative exponential. The sources have different modes of access to the N trunks, in two ways: (1) Source I has delay facilities, while II does not; its calls are lost if no trunk is available; (2) both I and II have delay, but I has priority over II. These cases are considered first under the assumption that the holding-times for I and II have the same average duration, and then also without this hypothesis, in which case the results are obtained only for a single trunk, $N=1$.

Equilibrium equations for the (birth and death) Markov process appropriate to each case are set up and solved for the distribution of queue lengths, the chances of loss and delay, the average delay, and, in the simplest case, delay distribution. When calls from one source are lost, the solutions are simple and resemble Erlang's original result on the delay problem for N trunks; but when priorities exist generating functions are required, and the solution depends on roots of (in this case quadratic) equations reminiscent of branching-process theory and the busy-time problem for queues.

V. E. Beneš.

★**Kramer, H. P.; and Mathews, M. V. A linear coding for transmitting a set of correlated signals.** Institute of Radio Engineers Transactions on Information Theory, IT-2, September 1956, pp. 41–46.

The authors consider the problem of optimal linear memoryless encoding and decoding of n stationary signals which are transmitted over m ($m\leq n$) noise-free channels. the ith signal be denoted by $e_i(t)$ and $R_{ij}=E\{e_i(t)e_j(t)\}$. The encoding and decoding operations are defined, respectively, by the transformations

$$x_\mu=\sum_{j=1}^{n} A_{\mu j}e_j \quad (\mu=1,\cdots,m),$$

$$e_i'=\sum_{\mu=1}^{m} B_{i\mu}x_\mu \quad (i=1,\cdots,n),$$

where the $A_{\mu j}$ and $B_{i\mu}$ are constants and the e_i' denote the received signals. The performance index of the system is the quantity $M=\sum_{i=1}^{n} E\{e_i-e_i'\}^2$. The authors show that M is minimized by making $B_{ij}=A_{ji}$ and setting A_{ij} equal to the jth component of the ith normalized eigenvector of the correlation matrix R. They also show that with this choice of the A_{ij} and B_{ij} the total channel capacity is reduced if the e_i signals are gaussian.

L. A. Zadeh (New York, N.Y.).

Control Systems

Skachkoff, B. N. On the stability of a class of non-linear systems of automatic regulation. Vestnik Leningrad. Univ. **12** (1957), no. 1, 46–56, 208. (Russian. English summary)

The author considers a control system characterized by the equations

$$(1)\qquad \begin{gathered}\dot\eta_1=a_{11}\eta_1+a_{12}\xi\\ \dot\eta_2=a_{21}\eta_2+a_{22}\xi\\ \dot\xi=f(\sigma),\ \ \sigma=a_{31}\eta_1+a_{32}\eta_2-\xi,\end{gathered}$$

where the a_{ij} are constants, $a_{11}<0$, $a_{21}<0$, and $f(\sigma)$ satisfies $f(\sigma)=0$ for $\sigma=0$ and $\sigma f(\sigma)>0$ for $|\sigma|>0$ in addition to the usual continuity and unicity assumptions. Lur'e [Some nonlinear problems of the theory of automatic regulation, Gostehizdat, Moscow-Leningrad, 1951; MR **15**, 707] has shown that under certain conditions on the parameters of (1) there exists a positive definite function

$$V=\int_0^\sigma f(\sigma)d\sigma+B_1\dot\eta_1^2+B_2\dot\eta_1\dot\eta_2+B_3\dot\eta_2^2,$$

where the last three terms represent a positive definite quadratic form, such that $\dot V$ is negative definite by virtue of (1). By using Lur'e's result and a theorem due to Pliss [Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 17–18; MR **17**, 152] the author obtains a set of sufficient conditions, too detailed to reproduce here, for the asymptotic stability in the large of the trivial solution of (1). *L. A. Zadeh.*

Popovici, Constantin P. Sur la théorie algébrique du clignoteur. Com. Acad. R. P. Romîne **6** (1956), 245–252. (Romanian. Russian and French summaries)

The blinker referred to in the title is one of various automatic mechanisms whose action has been described by Gr. C. Moisil [Acad. Repub. Pop. Romîne Bul. Şti. Sect. Şti. Mat. Fiz. **7** (1955), 183–230; MR **18**, 784] with the aid of Boolean algebra. The author gives a more direct way of writing down equations describing the system. In contrast to Moisil's the author's formulae involve additional physical variables, give the state at any instant in terms of the state at the preceding instant (instead of the two preceding instants), and are in disjunctive normal form in which each term is clearly interpretable. Equivalence with Moisil's formulae is however demonstrated. *I. M. H. Etherington* (Edinburgh).

Lee, C. Y.; and Chen, W. H. Several-valued combinational switching circuits. Trans. Amer. Inst. Electr. Eng., Part I, No. **25**, July 1956, pp. 278–283.

The algebras based upon the multivalued propositional logic of Post are applied to the design of switching circuits in which signals may assume any of n discrete levels or switches may assume any of n positions. This application is a generalization of the application of Boolean algebra to the design of two valued switching circuits.

It is found that the connectives and relationships which are commonly used in developing Post algebras are not the ones which are most convenient to use in the design of switching circuits. The authors take the three-valued case as an example and describe some general properties symmetric functions. A particular non-symmetric function of four variables called the T-operation is treated in detail. Any three-valued function may be represented as an expansion in terms of T-operation, in a manner which

permits the derivation of a corresponding circuit. The use of such expansions is illustrated in the design of a three-valued serial adder. Functions involved in this design are first represented in tabular form called the map representation. They are then expanded in terms of the T-operation and one other operation called the cycling operation. Finally, a circuit diagram is drawn in which gates replace the operations contained in the expansions.

D. E. Muller (Urbana, Ill.).

See also: Doetsch, p. 139; André, p. 142; Seibert, p. 142; Stein, p. 182.

HISTORY, BIOGRAPHY

McCrea, W. H. Obituary: Edmund Taylor Whittaker. J. London Math. Soc. 32 (1957), 234–256.

A biography written in support of the opening sentence "it may reasonably be claimed that no single individual in this century or the last had so far-reaching an influence upon the progress of British mathematics." There is a bibliography of Whittaker's published works, containing 13 books, 56 research papers, 35 philosophical and historical papers and 21 biographies.

Price, D. J. Obituary: Sir J. J. Thomson, O. M., F. R. S. Nuovo Cimento (10) 5 (1956), supplemento, 1609–1629.

See also: Barbour, p. 124; Steenrod, p. 158; Hodge, p. 173.

MISCELLANEOUS

Gnedenko, B. V. Mathematical education in the U.S.S.R. Amer. Math. Monthly **64** (1957), 389–408.

A description of all levels of mathematical activity. Many interesting problems are included, to be solved by secondary school students in their mathematical "Olympiads".

Cesari, Isotta. Residues of ideals (after a mathematicians' party). Amer. Math. Monthly **64** (1957), 420.

A humorous poem about conversation at a social gathering of mathematicians.

★ **Blaschke, Wilhelm. Reden und Reisen eines Geometers.** VEB Deutscher Verlag der Wissenschaften, Berlin, 1957. 118 pp. DM 7.20.

There are six lectures: Mathematik und Leben, Leonardo und die Naturwissenschaften, Kepler und Galilei (with a bibliography), Um die Welt, Regiomantus, Ein Lebenslauf (a short autobiography; the frontispiece is a photograph). These are popular lectures, mostly non-mathematical.

★ **Deaux, Roland. Introduction to the geometry of complex numbers.** Translated from the revised French edition by Howard Eves. Frederick Ungar Publishing Co., New York, 1957. 208 pp. $6.50.

The book, the result of years of lecturing on the subject to students of electro-mechanical engineering at the Faculté Polytechnique de Mons (Belgium), is a lucid elementary exposition of the subject and should prove useful to them as a "first book" before attempting more advanced ones on the same subject by J. L. Coolidge [The geometry of the complex domain, Oxford, 1924] or E. Cartan [Leçons sur la géométrie projective complexe, Gauthier-Villars, Paris, 1931]. Its first chapter interprets the geometric representation of complex numbers, the second comprises the essential concepts of plane analytic geometry in the form of equations between complex numbers, and the last chapter deals with the theory of circular transformations expressed in the same medium. Special attention is paid to cycloidal and unicursal curves, to homographies and antigraphies. The book, published by De Boeck, Bruxelles, 1947 in French, is enriched in this translation by over a hundred exercises, which add many essential theorems, together with helpful indications at solution. The author credits some technical improvements of proofs to the translator. The book is rather attractive as far as print, geometric figures and paper is concerned.

Saly R. Struik (Cambridge, Mass.).

★ **Günter, N. M.; und Kusmin, R. O. Aufgabensammlung zur höheren Mathematik. Bd. I.** VEB Deutscher Verlag der Wissenschaften, Berlin, 1957. viii+492 pp. DM 14.80.

A translation, by nine translators, based on the 10th through 12th editions of the Russian original [OGIZ, Moscow-Leningrad, 1945–1949].

The first volume contains problems up to the section on partial differential equations in the Russian original; the remaining sections are promised in a second volume. There are ten sections in the first volume, with an average of about 450 problems in each section, arranged approximately in order of difficulty. Solutions are given.

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